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Reduced phase space formalism for spherically symmetric geometry with a massive dust shell

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We perform a Hamiltonian reduction of spherically symmetric Einstein gravity with a thin dust shell of positive rest mass. Three spatial topologies are considered: Euclidean ($\mathbb{R}^3$), Kruskal ($S^2\times\mathbb{R}$), and the spatial topology of a diametrically identified Kruskal ($\mathbb{RP}^3\setminus$ {a point at infinity}). For the Kruskal and $\mathbb{RP}^3$ topologies the reduced phase space is four dimensional, with one canonical pair associated with the shell and the other with the geometry; the latter pair disappears if one prescribes the value of the Schwarzschild mass at an asymptopia or at a throat. For the Euclidean topology the reduced phase space is necessarily two dimensional, with only the canonical pair associated with the shell surviving. A time reparametrization on a two-dimensional phase space is introduced and used to bring the shell Hamiltonians to a simpler (and known) form associated with the proper time of the shell. An alternative reparametrization yields a square-root Hamiltonian that generalizes the Hamiltonian of a test shell in Minkowski space with respect to Minkowski time. Quantization is briefly discussed. The discrete mass spectrum that characterizes natural minisuperspace quantizations of vacuum wormholes and $\mathbb{RP}^3$ geons appears to persist as the geometrical part of the mass spectrum when the additional matter degree of freedom is added. [S0556-2821(97)00724-8]

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I. INTRODUCTION

In classical general relativity, every three-manifold occurs as the spatial topology of a globally hyperbolic vacuum spacetime. In a canonical approach to quantum gravity, the spatial topology is frozen, and one can ask for ground states corresponding to each topology.[1]

Spherically symmetric minisuperspaces provide simple models for the quantization of geometries with non-Euclidean topology. The spatial topologies consistent with spherical symmetry and asymptotic flatness are $\mathbb{R}^3$, the wormhole $S^2\times\mathbb{R}$ of the Kruskal geometry with two asymptopias, and the $\mathbb{RP}^3$ geon, a manifold with a single asymptopia obtained by removing a point from the compact manifold $\mathbb{RP}^3$. This last manifold is the space acquired from Kruskal geometry by identifying diametrically opposite points on an $U+V=$const slice, with U and V the usual Kruskal null coordinates [1].

A reduced phase space formalism for spherically symmetric vacuum Einstein gravity in four spacetime dimensions has been considered by several authors [2–12].[2] In the present paper we add to spherically symmetric Einstein gravity an idealized, infinitesimally thin dust shell of positive rest mass. The equations of motion for such a shell follow easily from Israel's junction condition formalism [32–35], and a number of workers have proposed actions from which these equations can be derived [36–42]. Our main purpose is to find an action for this system by an explicit Hamiltonian reduction, treating both the geometry and the shell as dynamical, and retaining the full dynamics allowed by the choice of the spatial topology.

Two issues require particular care. First, as general relativity is a nonlinear theory, introducing a distributional source faces well-known subtleties [34]. The special case of a source concentrated on a hypersurface of codimension 1 is fortunate, as Einstein's equations can then be given an unambiguous distributional interpretation, and this interpretation reduces to Israel's junction conditions when the source is a pure δ function on the surface [34]. However, we wish to

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[1]Even in a theory that permits topology change, topologies threaded by electric or magnetic flux in source-free Einstein-Maxwell theory (or in higher-dimensional gravity with Kaluza-Klein asymptotic behavior) cannot evolve to Euclidean space, if they have a net asymptotic charge. If there is a nonsingular quantum theory of such a system, it must allow a ground state with nonzero asymptotic charge and non-Euclidean topology. Topological geons with half-integral angular momentum in a quantum theory of gravity would similarly be unable to settle down to Euclidean topology.

[2]For extensions to related theories, including spherically symmetric Einstein-Maxwell theory and lower-dimensional dilatonic theories, see Refs. [10,13–21]. For discussions within the Euclidean context, see, for example, Refs. [22–31] and the references therein.

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go further and write an *action principle* from which the field equations would arise as variational equations. In such an action principle one presumably needs to be able to vary the action with respect to both the metric and the shell variables, with the variations remaining independent in some suitable sense. This brings in not only the regularity properties of the spacetime at the shell, but also the regularity properties of the coordinates in which the action is written.

We will not find an action principle whose variational equations would be fully distributionally consistent at the shell. However, the ambiguity in our variational equations will be localized into the single equation that results from varying the action with respect to the shell position coordinate. When the ambiguous contribution to this equation is interpreted as the average of its values on the two sides of the shell, as is necessitated by consistency with the rest of the equations, we correctly reproduce the content of Israel's junction condition formalism. At a somewhat formal level, our action will be manifestly invariant under the Hamiltonian version of spacetime coordinate transformations preserving spherical symmetry.

Second, one needs to choose the falloff and boundary conditions at the asymptopias. We shall set the asymptotic momenta to zero, but the values of the Schwarzschild masses at the asymptopias will be left free to emerge from the dynamics. Our spacelike hypersurfaces will not be asymptotic to hypersurfaces of constant Minkowski time, but the foliation is nevertheless asymptotically Minkowski in the relevant sense. In particular, the generator of unit time translations at the infinity is the Schwarzschild mass.

We shall find that the reduced phase space is four dimensional with the Kruskal and $\mathbb{RP}^3$ topologies, and two dimensional with the $\mathbb{R}^3$ topology. With each topology, one canonical pair is associated with the shell motion, but with the Kruskal and $\mathbb{RP}^3$ topologies there is also a second canonical pair, associated with the dynamics of the geometry. In the limit where the shell is removed, this reproduces results previously obtained in the Hamiltonian vacuum theories with the Kruskal and $\mathbb{RP}^3$ topologies [6,7].

For the non-Euclidean topologies, the canonical pair associated with the geometry disappears if one prescribes by hand the mass at one infinity in the Kruskal topology, and the mass at the wormhole throat in the $\mathbb{RP}^3$ topology. All three reduced phase spaces then become two dimensional, and they can be treated on an essentially equal footing.

We next introduce a formalism for reparametrizing time in a Hamiltonian theory with a two-dimensional phase space. Applying this formalism to our two-dimensional phase spaces, we redefine the coordinate time to coincide with the proper time of the shell and thereby obtain a Hamiltonian that can be given in terms of elementary functions. This Hamiltonian is known [37], but the fact that it emerges from a minisuperspace framework is new. An alternative choice for the coordinate time yields a Hamiltonian that generalizes to our self-gravitating shell the familiar $\sqrt{p^2+m^2}$ Hamiltonian of a spherical test shell in Minkowski space.

The paper concludes with a discussion of the prospects for quantization. Quantization of the vacuum case is revisited to emphasize choices that lead to discrete or continuous mass spectra. The additional degree of freedom provided by the shell does not appear to qualitatively alter these choices. Like the Jain-Schechter-Sorkin quantum-stabilized Skyrmion [43], the minisuperspace geons provide an example of field configurations that have quantum, but not classical ground states; both are field theory analogues of the quantum stabilization of the hydrogen atom. Whether the ground state of geons is an artifact of the reduction of the degrees of freedom is, of course, an open question, but the geometrical ground state appears to persist when the shell's degree of freedom is added.

With one asymptotic or interior mass fixed, the implicit Hamiltonian we obtain prior to time reparametrization was found by Kraus and Wilczek [44,45] in the limit of a massless shell, and it could easily be found from what they present also for the massive case. A related reduction technique was used earlier by Fischler *et al.* [46] in a minisuperspace treatment of a bubble wall, and recently generalized by Kolitch and Eardley [47]. For a flat geometry interior to the shell, our proper-time Hamiltonian has been considered classically in Ref. [36] and quantum mechanically in Ref. [38]. For a flat geometry interior to the shell, quantization using the square root Hamiltonian has been considered in Refs. [39,41].

Latin tensor indices $a,b,\dots$ indicate abstract spacetime indices. We work in Planck units, $\hbar=c=G=1$.

II. HAMILTONIAN FORMULATION FOR SPHERICALLY SYMMETRIC GEOMETRY WITH A DUST SHELL

In this section we set up a Hamiltonian formulation for spherically symmetric gravity coupled to a thin dust shell. We pay special attention to the smoothness of the gravitational variables and to the global boundary conditions.

A. Bulk action

A spherically symmetric spacetime metric can be locally written in the Arnowitt-Deser-Misner (ADM) form

$$ds^2=-N^2dt^2+\Lambda^2(dr+N^r dt)^2+R^2d\Omega^2, \qquad (2.1)$$

where $d\Omega^2$ is the metric on the unit two-sphere, and N, N^r, Λ, and R are functions of t and r. Issues of smoothness and global structure will be addressed below. We denote the derivative with respect to t by overdot, and the derivative with respect to r by prime.

The matter consists of a thin shell of dust, with a fixed positive rest mass m. We write the trajectory of the shell as $r=\mathfrak{r}(t)$. Denoting by $\hat{N}(t)$, $\hat{N}^r(t)$, $\hat{\Lambda}(t)$, and $\hat{R}(t)$ the values of N, N^r, Λ, and R at $r=\mathfrak{r}$,

$$\hat{R}(t):=R(t,\mathfrak{r}(t)), \quad \text{etc.}, \qquad (2.2)$$

the Hamiltonian action for the shell is [44,46]

$$S_{\text{shell}}=\int dt(\mathfrak{p}\dot{\mathfrak{r}}-\hat{N}\sqrt{\mathfrak{p}^2\hat{\Lambda}^{-2}+m^2}+\hat{N}^r\mathfrak{p}), \qquad (2.3)$$

with $\mathfrak{p}$ being the momentum conjugate to $\mathfrak{r}$. One can think of the shell as a spherically symmetric cloud of massive relativistic point particles.

The Lagrangian gravitational action for the geometry (2.1) is obtained by integrating the Lagrangian density

$(16\pi)^{-1}({}^3R - K^{ab}K_{ab} + K^2)\sqrt{-g}$ over the two-sphere [2,3,6,44,46,48]. After $\dot{\Lambda}$ and $\dot{R}$ are replaced by their conjugate momenta,

$$\pi_\Lambda = -\frac{R}{N}(\dot{R} - N^r R'), \tag{2.4a}$$

$$\pi_R = -\frac{\Lambda}{N}(\dot{R} - N^r R') - \frac{R}{N}[\dot{\Lambda} - (N^r\Lambda)'], \tag{2.4b}$$

the Hamiltonian bulk action for the coupled system reads

$$S_\Sigma = \int dt \left[\mathfrak{p}\dot{\mathfrak{r}} + \int dr(\pi_\Lambda \dot{\Lambda} + \pi_R \dot{R} - N\mathcal{H} - N^r \mathcal{H}_r) \right], \tag{2.5}$$

where the super-Hamiltonian $\mathcal{H}$ and the supermomentum $\mathcal{H}_r$ are given by

$$\mathcal{H} = \frac{\Lambda \pi_\Lambda^2}{2R^2} - \frac{\pi_\Lambda \pi_R}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} + \sqrt{\mathfrak{p}^2 \hat{\Lambda}^{-2} + m^2}\ \delta(r - \mathfrak{r}), \tag{2.6a}$$

$$\mathcal{H}_r = \pi_R R' - \pi_\Lambda' \Lambda - \mathfrak{p}\delta(r - \mathfrak{r}). \tag{2.6b}$$

We shall first discuss the smoothness of the gravitational variables, and then the boundary terms to be added to the bulk action.

B. Smoothness

In the presence of a smooth matter distribution, one can assume the spacetime metric to be smooth (C^∞). In the idealized case of an infinitesimally thin shell, the metric can be chosen to be continuous but not, in general, differentiable across the shell [32–35]. In the particular case of a spherically symmetric dust shell, Einstein's equations imply that the extrinsic curvature of the shell history is discontinuous both in its angular components and in its component along the shell four-velocity. If the metric is taken continuous, we must therefore accommodate discontinuities in R' and in at least some[3] of Λ', N', and $(N^r)'$. We would like both the action (2.5) and its local variations to be well defined, and such that the resulting variational equations are equivalent to Einstein's equations with a dust shell.

To proceed, we assume that the gravitational variables are smooth functions of r, with the exception that N', $(N^r)'$, Λ', R', π_Λ, and π_R may have finite discontinuities at isolated values of r, and that the coordinate loci of the discontinuities may be smooth functions of t. It will be shown that the resulting variational principle is satisfactory in the above sense, provided one of the variational equations is interpreted as the average of a discontinuous quantity over the two sides of the shell.[4] All the terms under the r integral in the action (2.5) are well defined in the distributional sense. The most singular contributions are the explicit matter δ contributions in the constraints, and the implicit δ functions in R'' and π_Λ'. All these δ functions are multiplied by continuous functions of r. The remaining terms are at worst discontinuous in r. The action is therefore well defined.

Local independent variations of the action with respect to the gravitational and matter variables give the constraint equations

$$\mathcal{H} = 0, \tag{2.7a}$$

$$\mathcal{H}_r = 0, \tag{2.7b}$$

and the dynamical equations

$$\dot{\Lambda} = N\left(\frac{\Lambda \pi_\Lambda}{R^2} - \frac{\pi_R}{R}\right) + (N^r\Lambda)', \tag{2.8a}$$

$$\dot{R} = -\frac{N\pi_\Lambda}{R} + N^r R', \tag{2.8b}$$

$$\dot{\pi}_\Lambda = \frac{N}{2}\left[-\frac{\pi_\Lambda^2}{R^2} - \left(\frac{R'}{\Lambda}\right)^2 + 1 + \frac{2\mathfrak{p}^2\delta(r-\mathfrak{r})}{\hat{\Lambda}^3\sqrt{\mathfrak{p}^2\hat{\Lambda}^{-2} + m^2}} \right] - \frac{N'RR'}{\Lambda^2} + N^r \pi_\Lambda', \tag{2.8c}$$

$$\dot{\pi}_R = N\left[\frac{\Lambda \pi_\Lambda^2}{R^3} - \frac{\pi_\Lambda \pi_R}{R^2} - \left(\frac{R'}{\Lambda}\right)' \right] - \left(\frac{N'R}{\Lambda}\right)' + (N^r \pi_R)', \tag{2.8d}$$

$$\dot{\mathfrak{r}} = \frac{\hat{N}\mathfrak{p}}{\hat{\Lambda}^2\sqrt{\mathfrak{p}^2\hat{\Lambda}^{-2} + m^2}} - \hat{N}^r, \tag{2.8e}$$

$$\dot{\mathfrak{p}} = \frac{\hat{N}\hat{\Lambda}'\mathfrak{p}^2}{\hat{\Lambda}^3\sqrt{\mathfrak{p}^2\hat{\Lambda}^{-2} + m^2}} - \hat{N}'\sqrt{\mathfrak{p}^2\hat{\Lambda}^{-2} + m^2} + \mathfrak{p}\widehat{(N^r)'}. \tag{2.8f}$$

With the exception of Eq. (2.8f), all the equations (2.7) and (2.8) are well defined in a distributional sense.[5] What needs

[3] By continuity of the metric, $\hat{R}(t)$ is well defined for all t. Taking the total time derivative of Eq. (2.2) shows that $\Delta\dot{R} = -\dot{\mathfrak{r}}\Delta R'$, where Δ denotes the discontinuity across the shell. Similarly for Λ, N, and N^r. Continuity of Λ', N', and $(N^r)'$ would therefore imply that the extrinsic curvature of the shell history is discontinuous only in its angular components.

[4] Because the constraint equations enforce smoothness of the metric outside the shell, our differentiability assumptions can probably be relaxed.

[5] The constraint Eqs. (2.7) contain explicit δ functions in r from the matter contribution and implicit δ functions in R'' and π_Λ'. The right-hand sides of Eqs. (2.8a) and (2.8b) contain at worst finite discontinuities, and the right-hand sides of Eqs. (2.8c) and (2.8d) contain at worst δ functions. This is consistent with the left-hand sides of Eqs. (2.8a)–(2.8d), because the loci of nonsmoothness in Λ, R, π_Λ and π_R may evolve smoothly in t. Note that both the explicit matter δ functions and the implicit δ functions in R'' and π_Λ' are multiplied by continuous functions of r.

to be examined is the consistency and dynamical content of the well-defined equations, and the interpretation of the single troublesome equation (2.8f).

As a preliminary, consider the variation of the matter action S_{shell} (2.3) with respect to the metric. From the definition of the stress-energy tensor,

$$\delta_g S_{\mathrm{shell}}=\frac{1}{2}\int \sqrt{-g}\; d^4x\; T^{ab}\;\delta(g_{ab}), \tag{2.9}$$

and the equation of motion (2.8e), we find that the surface stress-energy tensor of the shell [Ref. [33], Eq. (21.163)] takes the form

$$S^{ab}=\frac{m}{4\pi\hat{R}^2}\,u^a u^b, \tag{2.10}$$

where u^a is the four-velocity of the shell, normalized in the usual way $u^a u_a=-1$. This confirms that the shell indeed consists of pressureless dust, with surface energy density $m/(4\pi\hat{R}^2)$ and total rest mass m.

Also, recall that the full content of the Einstein equations at the shell is encoded in Israel's junction conditions [32,33]. We shall refer to the two sides of the shell as the ''right-hand side'' and the ''left-hand side,'' in view of the Penrose diagram in which two partial Kruskal diagrams are joined to each other along the shell trajectory. Israel's junction conditions then read

$$-8\pi(S_{ab}-\tfrac{1}{2}h_{ab}S)=K^{+}_{ab}-K^{-}_{ab}, \tag{2.11}$$

where n_a is the right-pointing unit normal to the shell history, $h_{ab}=g_{ab}-n_a n_b$ is the projector to this history, $K_{ab}=h^c{}_a h^d{}_b \nabla_c n_d$ is the extrinsic curvature tensor, and the signs $\pm$ refer, respectively, to the right and left sides of the shell. With Eq. (2.10), and with Kruskal geometries of masses $M_\pm$ on the two sides of the shell, the angular components of Eq. (2.11) read

$$-\frac{m}{\hat{R}}=\epsilon_+\sqrt{\left(\frac{d\hat{R}}{d\tau}\right)^2+1-\frac{2M_+}{\hat{R}}}-\epsilon_-\sqrt{\left(\frac{d\hat{R}}{d\tau}\right)^2+1-\frac{2M_-}{\hat{R}}}, \tag{2.12}$$

where τ is the shell's proper time. $\epsilon_+=1$ ($\epsilon_-=1$) if, when viewed from the geometry right (left) of the shell, the shell is in the right-hand-side exterior region of the Kruskal diagram, or if the shell is in the white-hole region and moving to the right, or if the shell is in the black-hole region and moving to the left. Otherwise $\epsilon_+=-1$ ($\epsilon_-=-1$). It can be verified that the shell motion is completely determined by the single equation (2.12) and the vacuum Einstein equations away from the shell. In particular, these equations imply that the tangential component of Eq. (2.11) is satisfied. A more explicit discussion can be found in Ref. [42].

Now, away from the shell, Eqs. (2.7) and (2.8) are well known to be equivalent to Einstein's equations. At the shell, the constraints (2.7) read

$$\Delta R'=-\frac{\sqrt{\mathfrak{p}^2+m^2\hat{\Lambda}^2}}{\hat{R}}, \tag{2.13a}$$

$$\Delta\pi_\Lambda=-\frac{\mathfrak{p}}{\hat{\Lambda}}, \tag{2.13b}$$

where we have adopted the notation (cf. footnote 3)

$$\Delta f:=\lim_{\epsilon\to 0_+}[f(\mathfrak{r}+\epsilon)-f(\mathfrak{r}-\epsilon)], \tag{2.14}$$

identifying $r>\mathfrak{r}$ ($r<\mathfrak{r}$) as the right (left) side of the shell. Using Eqs. (2.8e) and (2.10), one finds that Eq. (2.13a) is equivalent to Eq. (2.12). Our Hamiltonian equations away from the shell and the Hamiltonian constraint Eq. (2.7a) at the shell therefore form a system that is equivalent to the correct dynamics for the shell. When these equations hold, it can be verified that equation Eq. (2.13b) is proportional to Eq. (2.13a) by the nonsingular factor $\hat{R}(\dot{\mathfrak{r}}+\hat{N}^r)/\hat{N}$, and the momentum constraint thus contains no new information. Similarly, it can be verified that the δ parts in Eqs. (2.8c) and (2.8d) reduce to identities and contain no new information. Finally, Eqs. (2.8a) and (2.8b) contain no δ functions, and thus no new information, at the shell.

The single remaining equation of motion is Eq. (2.8f). If the geometry were smooth at the shell, Eqs. (2.8e) and (2.8f) would by construction be equivalent to the geodesic equation for the shell, as can indeed be explicitly verified. If the ambiguous spatial derivative terms in Eq. (2.8f) are evaluated on the left (right) side of the shell, Eq. (2.8f) thus implies the geodesic equation for the shell in the geometry on the left (right). However, these two geodesic equations are mutually inconsistent, and the shell motion implied by the rest of the equations is not geodesic in either of the two geometries. Instead, the rest of the equations imply that the left-hand side of Eq. (2.8f) is equal to the *average* of the right-hand side over the two sides of the shell. (For the generalization of this observation to nonspherical dust shells, see Exercise 21.26 in Ref. [33].) Therefore, if the ill-defined right-hand side of Eq. (2.8f) is given this average interpretation, our Hamiltonian formalism reproduces Einstein's equations for the dust shell.

We are not aware of an *a priori* justification of the averaged interpretation of the right-hand side of Eq. (2.8f). This interpretation is merely forced on us by the rest of the variational equations. In a strict sense, we therefore regard the variational principle as inconsistent, and the averaged interpretation of Eq. (2.8f) as put in by hand. Nevertheless, we shall proceed with this variational principle. It will be seen in Sec. III that the Hamiltonian reduction can be carried through with no apparent inconsistency.

One check on the consistency of the formalism is that the Poisson brackets of our constraints can be verified to obey the radial hypersurface deformation algebra [49], as in the absence of the shell. If we denote by $\mathcal{N}(r)$ and $\mathcal{N}^r(r)$ smooth smearing functions of compact support, the algebra has the form

$$\left\{\int dr\;\mathcal{N}_1\mathcal{H},\int dr\;\mathcal{N}_2\mathcal{H}\right\}=\int dr\;(\mathcal{N}_1\mathcal{N}_2'-\mathcal{N}_2\mathcal{N}_1')\Lambda^{-2}\mathcal{H}_r, \tag{2.15a}$$

$$\left\{\int dr\;\mathcal{N}^r\mathcal{H}_r,\int dr\;\mathcal{N}\mathcal{H}\right\}=\int dr\;\mathcal{N}^r\mathcal{N}'\mathcal{H}, \tag{2.15b}$$

$$\left\{\int dr\, \mathcal{N}_1^r \mathcal{H}_r, \int dr\, \mathcal{N}_2^r \mathcal{H}_r\right\} = \int dr\, [\mathcal{N}_1^r (\mathcal{N}_2^r)' - \mathcal{N}_2^r (\mathcal{N}_1^r)']\mathcal{H}_r. \tag{2.15c}$$

C. Asymptopias and boundary terms

We now turn to the global properties of the geometry. In this section we take the spatial topology to be that of the extended Schwarzschild geometry, $S^2 \times \mathbb{R} = S^3 \setminus \{\text{two points}\}$, the omitted points being associated with asymptotically flat asymptopias. The spatial topologies $\mathbb{RP}^3 \setminus \{\text{a point at infinity}\}$ and $\mathbb{R}^3$ will be discussed, respectively, in Secs. V and VI.

At a general level, restricting the asymptotic behavior of an asymptotically flat system allows one to fix the momentum, angular momentum, and mass at spatial infinity. In a quantum theoretic context, to restrict in this way the asymptotic behavior of the operator ${}^3\hat{g}_{ab}$ and its conjugate momentum $\hat{\pi}^{ab}$ is equivalent to restricting the state space to an eigensubspace of fixed total momentum, angular momentum, or mass. In our particular case of spherical symmetry, the angular momentum is necessarily zero. It would be consistent with spherical symmetry to allow a nonzero momentum at infinity (in the classical framework, this would mean allowing boosted Schwarzschild solutions), but for our purposes this freedom does not appear significant, and we shall set the momentum at infinity to zero. We shall, however, retain the freedom associated with the system's total mass.

We take the coordinate r to have the range $-\infty < r < \infty$. At the asymptopias $r \to \pm\infty$, we introduce the falloff

$$\Lambda(t,r) = 1 + O^\infty(|r|^{-3/2-\beta}), \tag{2.16a}$$

$$R(t,r) = |r| + O^\infty(|r|^{-1/2-\beta}), \tag{2.16b}$$

$$\pi_\Lambda(t,r) = \sqrt{2M_\pm |r|} + O^\infty(|r|^{-\beta}), \tag{2.16c}$$

$$\pi_R(t,r) = \sqrt{\frac{M_\pm}{2|r|}} + O^\infty(|r|^{-1-\beta}), \tag{2.16d}$$

$$N(t,r) = 1 + O^\infty(|r|^{-\beta}), \tag{2.16e}$$

$$N^r(t,r) = \pm\sqrt{\frac{2M_\pm}{|r|}} + O^\infty(|r|^{-1/2-\beta}), \tag{2.16f}$$

where $M_\pm(t)$ are positive-valued functions of t, and β is a positive parameter that can be chosen at will. O^∞ indicates a quantity that is bounded at infinity by a constant times its argument, with the corresponding behavior for its derivatives.

It is straightforward to verify that the falloff (2.16) is consistent with the constraints and preserved in time by the dynamical equations. When the equations of motion hold, $M_\pm$ are independent of t, and their values are just the Schwarzschild masses at the two asymptopias. Using Eq. (2.12), it is easy to show that the existence of two asymptotically flat infinities implies that both asymptotic Schwarzschild masses in the classical solutions are necessarily positive. The assumption $M_\pm(t) > 0$ in Eq. (2.16) does therefore not exclude any solutions.

The falloff (2.16) is not consistent with the conventional falloffs (see, for example, Refs. [6,50]) in which the hypersurfaces of constant t are asymptotic to hypersurfaces of constant Killing time when the equations of motion hold. Instead, the falloff (2.16) is asymptotic to the ingoing spatially flat coordinates [51–53], individually near each asymptopia. When $M_\pm$ are constants and all the O^∞-terms vanish, Eq. (2.16) yields the Schwarzschild metric in the ingoing spatially flat coordinates, separately for $r > 0$ and $r < 0$. Our reason for adopting Eq. (2.16) is that the spatially flat gauge will prove useful in the Hamiltonian reduction in Sec. III [44].

In a variational principle that does not fix the values of $M_\pm$, the bulk action (2.5) must be amended by a boundary action. With our falloff (2.16), the spatial metric approaches flatness at $r \to \pm\infty$ so fast that the variations of R and Λ give rise to no boundary terms from the infinities. The only nontrivial boundary term arises from integrating by parts the term $\int dt \int dr\, N^r \Lambda (\delta\pi_\Lambda)'$, associated with the momentum constraint. This boundary term is canceled if we add to the bulk action (2.5) the boundary action

$$S_{\partial\Sigma} = -\int dt\, (M_+ + M_-). \tag{2.17}$$

The generator of unit time translations at the infinities is therefore still the Schwarzschild mass, despite the unconventional falloff.

III. REDUCED PHASE SPACE FORMULATION

In the absence of the shell, the Hamiltonian reduction of our theory with a technically different but qualitatively similar falloff at the two asymptopias was discussed in Ref. [6]. When the asymptotic masses are not fixed, it was found that the reduced phase space is two dimensional, whereas if one asymptotic mass is fixed, the reduced phase space has dimension zero. As the shell brings in one new canonical pair but no new constraints, one expects that the reduced phase space of our theory is four dimensional when the asymptotic masses are not fixed, and two dimensional if one asymptotic mass is fixed. In this section we shall verify this expectation by an explicit Hamiltonian reduction.

A. Gauge transformations and the Hamiltonian reduction formalism

In the Hamiltonian theory formulated in Sec. II, the variables $(\Lambda, R, \pi_\Lambda, \pi_R, \mathfrak{r}, \mathfrak{p})$ constitute a canonical chart on the phase space $\mathcal{S}$, while N and N^r act as Lagrange multipliers enforcing the constraints. As the Poisson bracket algebra (2.15) of the constraints closes, we have a first class constrained system [54].

Let Γ denote the constraint hypersurface (2.7) in $\mathcal{S}$. We take gauge transformations to mean the transformations on Γ

generated by the constraints.[6] Denoting the smearing functions by $\mathcal{N}(r)$ and $\mathcal{N}^r(r)$ as in Eq. (2.15), the smeared Hamiltonian constraint transforms an initial data set $(\Lambda,R,\pi_\Lambda,\pi_R,\mathfrak{r},\mathfrak{p})\in\Gamma$ by the time evolution associated with $\mathcal{N}$, and the smeared momentum constraint transforms the initial data set by the spatial diffeomorphism associated with $\mathcal{N}^r$. The smearing functions must fall off so fast that the transformations become trivial at the infinities and the falloff (2.16) is preserved.[7]

By definition, the reduced phase space $\overline{\Gamma}$ consists of the equivalence classes in Γ under gauge transformations. The symplectic form ω on $\mathcal{S}$,

$$\omega:=\boldsymbol{\delta}\mathfrak{p}\wedge\boldsymbol{\delta}\mathfrak{r}+\int dr(\boldsymbol{\delta}\pi_\Lambda\wedge\boldsymbol{\delta}\Lambda+\boldsymbol{\delta}\pi_R\wedge\boldsymbol{\delta}R), \qquad (3.1)$$

induces a symplectic form $\hat{\omega}$ on $\overline{\Gamma}$. Here, and from now on, $\boldsymbol{\delta}$ denotes the exterior derivative on the (functional) spaces in question.

We wish to implement this Hamiltonian reduction, finding $\hat{\omega}$ in an explicit symplectic chart on $\overline{\Gamma}$. Our implementation will consist of the following three steps.

(1) Consider first Γ. At the shell, we have already seen that the full content of the constraints is encoded in Eqs. (2.13). Away from the shell, the constraints can be solved explicitly for the gravitational momenta as [44,46]

$$\pi_\Lambda=R\sqrt{(R'/\Lambda)^2-1+2M_\pm/R}, \qquad (3.2a)$$

$$\pi_R=\frac{\Lambda[(R/\Lambda)(R'/\Lambda)'+(R'/\Lambda)^2-1+M_\pm/R]}{\sqrt{(R'/\Lambda)^2-1+2M_\pm/R}}, \qquad (3.2b)$$

with the upper (lower) signs holding respectively for $r>\mathfrak{r}$ $(r<\mathfrak{r})$. We have chosen the sign of the square root in Eq. (3.2) so as to agree with the falloff (2.16). This choice will lead to a reduction that will cover the black hole interior but not the white hole interior.

(2) To pass from Γ to $\overline{\Gamma}$, we choose a gauge: we specify in Γ a hypersurface $\overline{H}$ that is transversal to the gauge orbits, so that each point in (an open subset of) $\overline{\Gamma}$ has a unique representative in $\overline{H}$. This defines an isomorphism between $\overline{H}$ and (the open subset of) $\overline{\Gamma}$. In order to choose the gauge in practice, we note that away from the shell, a point $(\Lambda,R,\pi_\Lambda,\pi_R,\mathfrak{r},\mathfrak{p})\in\Gamma$ is an initial data set for the *vacuum* Einstein equations with spherical symmetry. Any vacuum initial data set has a unique time evolution, and, by Birkhoff's theorem, the resulting subspacetimes left and right of the shell are isometric to regions of two Kruskal spacetimes with the respective masses M_- and M_+. A solution to the constraint equations can thus be regarded as two parametrized partial spacelike hypersurfaces in the two Kruskal spacetimes, joining appropriately at the shell. In this picture, a gauge choice means making a particular choice for these two partial spacelike hypersurfaces in the two Kruskal spacetimes, in a way that joins appropriately at the shell and is compatible with the falloff at the infinities.

(3) To find the symplectic form $\hat{\omega}$ on $\overline{\Gamma}$, it is convenient first to find the corresponding Liouville form. Recall that on $\mathcal{S}$, the Liouville form corresponding to our canonical chart $(\Lambda,R,\pi_\Lambda,\pi_R,\mathfrak{r},\mathfrak{p})$ is

$$\theta:=\mathfrak{p}\boldsymbol{\delta}\mathfrak{r}+\int dr(\pi_\Lambda\boldsymbol{\delta}\Lambda+\pi_R\boldsymbol{\delta}R). \qquad (3.3)$$

Pulling θ back to $\overline{H}$ yields on $\overline{H}$ the Liouville form $\hat{\theta}_{\overline{H}}$, and $\hat{\omega}_{\overline{H}}:=\boldsymbol{\delta}\hat{\theta}_{\overline{H}}$ is the symplectic form on $\overline{H}$ that corresponds to $\hat{\omega}$ on (the isomorphic open subset of) $\overline{\Gamma}$.

In view of the description of the gauge choice in step (2), a technical point in step (3) arises from the fact that although $M_\pm$ are constants in the time evolution of a given initial data set, they are not constants as functions on $\overline{H}$, and their exterior derivatives may contribute to the pullback of θ (3.3). Put differently, a generic path in $\overline{\Gamma}$ need not correspond to a partial foliation of a single Kruskal geometry on either side of the shell.

To complete steps (2) and (3), we need to specify the gauge. This will be described next.

B. Gauge choice

Our gauge choice involves taking the intrinsic metric on the spacelike hypersurface to be flat, with the exception of certain transition regions that are eventually taken to be vanishingly narrow. The possible locations for the transition regions depend on whether the shell trajectory is visible to the right-hand-side future null infinity, the left-hand-side future null infinity, or neither.[8] We now make the simplifying assumption that part of the shell trajectory is visible to one future null infinity, and we take this infinity to be on the right. This is arguably the situation of physical interest for an observer in the asymptotically flat region.

Thus, fix an initial data set (a point in Γ), and consider the classical spacetime that is its time evolution. We assume that in this spacetime, the shell trajectory intersects the right-hand-side exterior region in the Kruskal geometry right of the shell. The shell equation of motion (2.12) then implies $M_+>M_-$, and the trajectory intersects the right-hand-side exterior region also in the Kruskal geometry left of the shell. It follows that $\epsilon_-=1$ on all of the trajectory, whereas $\epsilon_+=1$ when $\hat{R}$ is sufficiently large (in particular, when $\hat{R}>2M_+$) but $\epsilon_+=-1$ as $\hat{R}\to 0$.

[6]See, for example, Ref. [55]. Note that this is distinct from, although closely related to, the gauge transformations that act on the histories on which the action is defined [54,56,57].

[7]One could consider an extended phase space that contains N and N^r as new coordinates and their conjugates π_N and π_{N^r} as new momenta. We shall, however, not need this extension.

[8]This last case occurs when $\epsilon_+=-1$ and $\epsilon_-=1$ in Eq. (2.12). The spacetime has two bifurcation spheres, and the shell passes between them, remaining at all times behind the white-hole and black-hole horizons of each infinity [58]. We are grateful to Eric Poisson for discussions on this case.

On this spacetime, we introduce two local charts, C_1 and C_2, as follows.

Suppressing the angles, let the coordinates in the chart C_1 be (t_1,r_1), with $r_1>0$. The metric reads

$$ds^2=-dt_1^2+\left(dr_1+\sqrt{\frac{2M_-}{r_1}}\,dt_1\right)^2+r_1^2d\Omega^2,\qquad 0<r_1\leqslant\mathfrak{r}-l, \tag{3.4a}$$

$$ds^2=-dt_1^2+\left(dr_1+\sqrt{\frac{2M_+}{r_1}}\,dt_1\right)^2+r_1^2d\Omega^2,\quad \mathfrak{r}\leqslant r_1, \tag{3.4b}$$

where l is a positive parameter. The two metrics shown in Eq. (3.4) are the ingoing right-hand-side spatially flat charts in Kruskal manifolds with the respective masses M_- and M_+ [51–53]. If taken individually for $0<r_1<\infty$ and $-\infty<t_1<\infty$, each of these two metrics would cover the upper right half (that is, the right-hand-side exterior and the black hole interior) in the respective full Kruskal diagrams. With the domains indicated in Eq. (3.4), the combined chart is spatially flat with mass M_- for $r_1\leqslant\mathfrak{r}-l$, and spatially flat with mass M_+ for $r_1\geqslant\mathfrak{r}$. The chart in the transition region $\mathfrak{r}-l\leqslant r_1\leqslant\mathfrak{r}$ will be specified below.

Let the coordinates in the chart C_2 be (t_2,r_2), with $r_2<0$. The metric reads

$$ds^2=-dt_2^2+\left(-dr_2+\sqrt{\frac{2M_-}{|r_2|}}\,dt_2\right)^2+r_2^2d\Omega^2,\quad r_2<0. \tag{3.5}$$

We identify C_2 as the ingoing left-hand-side spatially flat chart in a Kruskal manifold with mass M_-, with $r_2\to-\infty$ giving the infinity on the left. If $-\infty<t_2<\infty$, the metric (3.5) covers the upper left half (that is, the left-hand-side exterior and the black hole interior) in the Penrose diagram of this Kruskal manifold. On our spacetime, C_2 covers the corresponding regions left of the shell.

Now, consider our initial data set as a parametrized spacelike hypersurface Σ_0 in this spacetime. By the falloff (2.16), Σ_0 is asymptotic at $r\to\infty$ to a constant t_1 hypersurface Σ_1 in the chart C_1, with r being asymptotic to r_1. Similarly, Σ_0 is asymptotic at $r\to-\infty$ to a constant t_2 hypersurface Σ_2 in the chart C_2, with r being asymptotic to r_2. Without loss of generality, we can take Σ_1 and Σ_2 to be, respectively, the hypersurfaces $t_1=0$ and $t_2=0$. We assume that Σ_1 and Σ_2 intersect, and that they do so left of the shell, in the black hole interior in the left-hand-side Kruskal geometry.[9] The value of R at the intersection (where $R=r_1=-r_2$) is denoted by ρ. Note that ρ can be regarded as a piece of gauge-invariant information in our initial data set.

Let $\hat{\Sigma}_0$ be the hypersurface consisting of Σ_1 for $r_1\geqslant\rho$ and Σ_2 for $r_2\leqslant-\rho$. $\hat{\Sigma}_0$ is not smooth, but has a corner (a sharp ridge) at $r_1=-r_2=\rho$. We choose a positive parameter γ, and we take γ and l so small that $(1+\gamma)\rho<\min(2M_-,\mathfrak{r}-l)$. We now deform $\hat{\Sigma}_0$ in the regions $-(1+\gamma)\rho\leqslant r_2\leqslant-\rho$ and $\rho\leqslant r_1\leqslant(1+\gamma)\rho$, in a way specified below, so that the deformed hypersurface $\tilde{\Sigma}_0$ becomes a smooth, parametrized hypersurface, with a parameter r that coincides with r_1 for $r\geqslant(1+\gamma)\rho$ and with r_2 for $r\leqslant-(1+\gamma)\rho$. The canonical data on $\tilde{\Sigma}_0$ is by construction gauge equivalent to our original initial data, and it becomes uniquely determined after we specify $\tilde{\Sigma}_0$ in the transition regions $|r|\leqslant(1+\gamma)\rho$ and $\mathfrak{r}-l\leqslant r\leqslant\mathfrak{r}$. As our gauge choice, we take canonical data on $\tilde{\Sigma}_0$ as the representative from the gauge equivalence class of our original initial data.

C. Liouville form and the reduced Hamiltonian theory

We now find the Liouville form $\hat{\theta}_{\overline{H}}$ by pulling the Liouville form θ (3.3) back to the transversal surface $\overline{H}$. This means that we need to evaluate the right-hand side of Eq. (3.3) when the constraints and our gauge condition hold.

Outside the transition regions $|r|\leqslant(1+\gamma)\rho$ and $\mathfrak{r}-l\leqslant r\leqslant\mathfrak{r}$, our gauge reads

$$R(r)=|r|, \tag{3.6a}$$

$$\Lambda(r)=1, \tag{3.6b}$$

with the gravitational momenta given by Eq. (3.2). With Eq. (3.6), $\boldsymbol{\delta}R$ and $\boldsymbol{\delta}\Lambda$ vanish. The only contributions to the integral in Eq. (3.3) therefore come from the transition regions. We evaluate these contributions in Appendices A and B, specifying the gauge in the transition regions and finally passing to the limit where the parameters l and γ vanish. From Eqs. (A11) and (B7), we find

$$\hat{\theta}_{\overline{H}}=p_\rho\boldsymbol{\delta}\rho+p\,\boldsymbol{\delta}\mathfrak{r}, \tag{3.7}$$

where

$$p_\rho:=\rho\,\ln\left(\frac{\sqrt{2M_-}+\sqrt{\rho}}{\sqrt{2M_-}-\sqrt{\rho}}\right)-2\sqrt{2M_-\rho}, \tag{3.8a}$$

$$p:=\sqrt{2M_-\mathfrak{r}}-\sqrt{2M_+\mathfrak{r}}+\mathfrak{r}\,\ln\left(\frac{\mathfrak{r}+\mathfrak{p}+\sqrt{\mathfrak{p}^2+m^2}+\sqrt{2M_+\mathfrak{r}}}{\mathfrak{r}+\sqrt{2M_-\mathfrak{r}}}\right), \tag{3.8b}$$

with $\mathfrak{p}$ being a solution to

$$M_+-M_-=\sqrt{\mathfrak{p}^2+m^2}+\frac{m^2}{2\mathfrak{r}}-\mathfrak{p}\sqrt{\frac{2M_+}{\mathfrak{r}}}. \tag{3.9}$$

Equation (3.9) has been obtained by eliminating R'_- from Eqs. (B3) and (B4).

The reduction is thus complete. The functions $(\rho,p_\rho,\mathfrak{r},p)$ provide a local canonical chart on the reduced phase space $\overline{\Gamma}$, and Eqs. (3.8) and (3.9) determine M_+ and M_- as functions in this canonical chart. The Hamiltonian, read off from Eq. (2.17), is

[9] This assumption is a further restriction on the initial data. Qualitatively, it tells how ''early'' or ''late'' the asymptotic ends of Σ_0 may be with respect to each other and the shell trajectory.

$$h := M_+ + M_- , \tag{3.10}$$

and the reduced action reads

$$S = \int dt \, (p_\rho \dot{\rho} + p \dot{\mathfrak{r}} - h). \tag{3.11}$$

As anticipated, $\overline{\Gamma}$ has dimension four.

D. Dynamics in the reduced theory

For understanding the dynamical content of the reduced theory, it is useful to introduce the new canonical chart $(M_-, P_-, \mathfrak{r}, p)$, defined by Eq. (3.8a) and

$$P_- := 4M_- \ln\left(\frac{\sqrt{2M_-} + \sqrt{\rho}}{\sqrt{2M_-} - \sqrt{\rho}} \right) - 4\sqrt{2M_- \rho}. \tag{3.12}$$

The new action reads

$$S = \int dt \, (P_- \dot{M}_- + p \dot{\mathfrak{r}} - h), \tag{3.13}$$

where the Hamiltonian $h(\mathfrak{r}, p, M_-)$ is determined by Eqs. (3.8b)–(3.10). In this chart, it is immediate that both M_- and M_+ are constants of motion. It is straightforward to verify that the equations of motion for the shell variables are equivalent to Eq. (2.12), and thus yield the correct dynamics, provided t is identified as the coordinate t_1 in the spatially flat chart (3.4b) right of the shell. The two solutions of Eq. (3.9) for $\mathfrak{p}$ correspond to $\epsilon_+ = \pm 1$ in Eq. (2.12), whereas $\epsilon_- = 1$ always by virtue of the global assumptions made above. We shall provide the key steps of this calculation below in Sec. IV.

What remains is the spacetime interpretation of the variable ρ. Recall that on the initial data hypersurface Σ_0 introduced in Sec. III B, ρ is the value of R at the sharp ridge where the hypersurface $t_1 = 0$ in the chart C_1 (3.4), asymptotic to Σ_0 at $r \to \infty$, meets the hypersurface $t_2 = 0$ in the chart C_2 (3.5), asymptotic to Σ_0 at $r \to -\infty$. Recall also that our Hamiltonian evolves the spacelike hypersurfaces so that at the two infinities, covered, respectively, by the charts C_1 and C_2, we have $dt_1/dt = 1$ and $dt_2/dt = 1$. One might therefore have thought that as our initial data evolves, $\rho(t)$ would be the value of R at the sharp ridge where the hypersurface $t_1 = t$ in the chart (3.4a) meets the hypersurface $t_2 = t$ in the chart (3.5). However, this does not hold. The reason is that in the $l \to 0$ limit, the chart C_1 does not reduce to a consistent chart across the shell, not even if one were to allow nondifferentiability: the intrinsic metric on the shell history is unambiguous, but evaluating this intrinsic metric from the $l \to 0$ limit of Eq. (3.4a) and from Eq. (3.4b) leads to mutually inconsistent expressions because the two masses differ. This means that if one approaches the shell from the two sides on the ''same'' constant t_1 hypersurface, after having first taken the limit $l \to 0$, one arrives at two different two spheres on the shell history. The $l \to 0$ limit of *one* constant t_1 hypersurface can be interpreted as a continuous hypersurface in the spacetime, and this is what we utilized in the gauge choice and the evaluation of the Liouville form, but one cannot maintain such an interpretation for a full foliation where t_1 takes values in an open interval. The spacetime interpretation of the variable ρ must therefore be examined more carefully.

Consider the chart $\widetilde{C}_1^-$ obtained as the $l \to 0$ limit of the chart C_1 left of the shell. Denoting the coordinates in $\widetilde{C}_1^-$ by $(\tilde{t}_1, r_1)$, the metric reads

$$ds^2 = -d\tilde{t}_1^2 + \left(dr_1 + \sqrt{\frac{2M_-}{r_1}} \, d\tilde{t}_1 \right)^2 + r_1^2 d\Omega^2, \quad 0 < r_1 \leqslant \mathfrak{r}. \tag{3.14}$$

If τ is the proper time along the shell history, we have from Eqs. (3.4b) and (3.14) the relation

$$\begin{aligned} d\tau^2 &= dt_1^2 - \left(d\mathfrak{r} + \sqrt{\frac{2M_+}{\mathfrak{r}}} \, dt_1 \right)^2 \\ &= d\tilde{t}_1^2 - \left(d\mathfrak{r} + \sqrt{\frac{2M_-}{\mathfrak{r}}} \, d\tilde{t}_1 \right)^2 . \end{aligned} \tag{3.15}$$

If we fix the hypersurface $\tilde{t}_1 = 0$ to coincide with the $l \to 0$ limit of the initial hypersurface $t_1 = 0$ for $0 < r_1 \leqslant \mathfrak{r}$, the relation (3.15) determines $\tilde{t}_1$ as a function of t_1 and the shell motion, $\tilde{t}_1 = \hat{t}_1(t_1)$. It can now be verified that $\rho(t)$ is the value of R at the sharp ridge where the hypersurface $\tilde{t}_1 = \hat{t}_1(t)$ in the chart (3.14) meets the hypersurface $t_2 = t$ in the chart (3.5). The algebra involved in this calculation appears not to be particularly instructive, and it will not be reproduced here.

E. Comments

As noted above, the details of our reduction relied on certain qualitative assumptions about the shell motion. In particular, we assumed the shell trajectory to intersect the right-hand-side exterior region of the Kruskal geometry right of the shell. Our gauge choice, involving the *ingoing* spatially flat coordinates, allows us to follow the shell trajectories into the black hole, but not into the white hole. A time-reversed gauge choice, involving the *outgoing* spatially flat coordinates, would conversely allow us to follow the trajectories into the white hole but not into the black hole.

In the reduced theory (3.13), the value of the canonical coordinate M_- is a constant of motion. If we are only interested in the shell motion, we can reduce the theory further by dropping the Liouville term $P_- \dot{M}_-$ and regarding M_- as a prescribed positive constant. This is arguably the theory of physical interest for an observer who scrutinizes the shell motion from one asymptotically flat infinity and regards the ''interior'' mass as fixed. The action then reads

$$S = \int dt \, (p \dot{\mathfrak{r}} - h), \tag{3.16}$$

where $h(\mathfrak{r}, p, M_-)$ is determined by Eqs. (3.8b)–(3.10). In the limit $m \to 0$, this theory reduces to that obtained by Kraus and Wilczek [44] by a less direct Hamiltonian reduction.

IV. TIME REPARAMETRIZATION

In this section we first present a general formalism for reparametrizing time in a Hamiltonian system with a two-dimensional phase space. We then apply this formalism to the reduced Hamiltonian theory (3.16).

A. General time-reparametrization formalism for two-dimensional phase space

Consider a Hamiltonian system with a two-dimensional phase space $\mathbf{\Gamma} := \{(q,p)\}$ and a time-independent Hamiltonian $h(q,p)$. With respect to a time t, Hamilton's equations read

$$\frac{dq}{dt} = \frac{\partial h}{\partial p}, \tag{4.1a}$$

$$\frac{dp}{dt} = -\frac{\partial h}{\partial q}. \tag{4.1b}$$

We wish to find a Hamiltonian system that generates the equivalent dynamics with respect to a new parameter time T, related to t by

$$dT = N dt, \tag{4.2}$$

where N is a prescribed (positive) function of some suitable set of dynamical variables. We further wish this time reparametrization to preserve the value of the Hamiltonian for each solution to the equations of motion (4.1). We examine separately two cases: (1) N is a function on $\mathbf{\Gamma}$, and (2) N is a function of q and the new velocity $V := dq/dT$.

1. $N = N(q,p)$

Suppose that $N(q,p)$ is a prescribed function on $\mathbf{\Gamma}$. We replace p by a new momentum $P := \hat{P}(q,p)$, where

$$\frac{\partial \hat{P}(q,p)}{\partial p} = N(q,p). \tag{4.3}$$

We assume that $\hat{P}(q,p)$ is an invertible function of p for each q, with the inverse $\hat{p}(q,P)$. The new phase space is $\hat{\mathbf{\Gamma}} := \{(q,P)\}$, and we take the Hamiltonian on $\hat{\mathbf{\Gamma}}$ to be

$$H(q,P) := h(q, \hat{p}(q,P)). \tag{4.4}$$

Hamilton's equations on $\hat{\mathbf{\Gamma}}$ with respect to a time T are then easily seen to be equivalent to Eq. (4.1), provided t and T are related by Eq. (4.2).

2. $N = N(q,V)$

Suppose next that $N(q,V)$ is a prescribed function of q and the new velocity V.

Recall that Eq. (4.1a) defines the velocity $v := dq/dt$ as a function on $\mathbf{\Gamma}$. We assume that this function can be inverted for the momentum as $p = \tilde{p}(q,v)$. We can then define on the velocity space the energy function

$$\tilde{h}(q,v) := h(q, \tilde{p}(q,v)). \tag{4.5}$$

The dynamics is now encoded in the statement that $\tilde{h}(q,v)$ is constant in t. The value of $\tilde{h}(q,v)$ provides one constant of integration, and expressing dq/dt in terms of this constant and q yields the general solution in terms of a single quadrature.

Consider now the time reparametrization (4.2) with $N = N(q,V)$. The velocities $v = dq/dt$ and $V = dq/dT$ are related by

$$v = N(q,V)V. \tag{4.6}$$

Using Eq. (4.6), we can define on the *new* velocity space the energy function

$$\tilde{H}(q,V) := \tilde{h}(q, N(q,V)V). \tag{4.7}$$

Provided the relation (4.6) between the velocities is not degenerate, the full dynamics is then encoded in the statement that $\tilde{H}(q,V)$ is constant in T.

We wish to find a Hamiltonian $H(q,P)$ from which $\tilde{H}(q,V)$ emerges as the energy function. If $L(q,V)$ is the corresponding Lagrangian, we have

$$\tilde{H}(q,V) = V \frac{\partial L(q,V)}{\partial V} - L(q,V) \tag{4.8}$$

and

$$P(q,V) = \frac{\partial L(q,V)}{\partial V}. \tag{4.9}$$

Solving Eq. (4.8) for $L(q,V)$, we find from Eq. (4.9) that the general solution for $P(q,V)$ is equivalent to

$$\frac{\partial P(q,V)}{\partial V} = V^{-1} \frac{\partial \tilde{H}(q,V)}{\partial V}. \tag{4.10}$$

The Hamiltonian $H(q,P)$ is obtained by inverting $P(q,V)$ for V and substituting this in $\tilde{H}(q,V)$.

3. Comments

Our time reparametrization preserves the value of the Hamiltonian on each solution to the equations of motion. It does not, however, preserve the value of the action, and it cannot in general be thought of as a canonical transformation.

After N is specified, the solutions to Eqs. (4.3) and (4.10) each contain an arbitrary additive function of q. This arbitrariness corresponds to a canonical transformation that redefines P by the addition of (the gradient of) an arbitrary function.

4. Example: relativistic particle in (1+1)-dimensional Minkowski space

As a simple example, we apply this reparametrization formalism to the free relativistic particle in $(1+1)$-dimensional Minkowski space.

We start from the Hamiltonian

$$h=\sqrt{p^2+m^2}, \tag{4.11}$$

which evolves the particle in the Minkowski time t. We then have

$$v=\frac{\partial h}{\partial p}=\frac{p}{\sqrt{p^2+m^2}}. \tag{4.12}$$

We wish to identify the new time parameter T as the proper time of the particle. From Eqs. (4.2) and (4.12) we then obtain

$$N=\sqrt{1-v^2}=\frac{m}{\sqrt{p^2+m^2}}. \tag{4.13}$$

We can thus use the above formalism with $N(q,p)=m(p^2+m^2)^{-1/2}$. As a solution to Eq. (4.3), we choose $\hat{P}(q,p)=m\ \mathrm{arcsinh}(p/m)$. This leads to the familiar point particle proper time Hamiltonian

$$H(q,P)=m\ \cosh(P/m). \tag{4.14}$$

B. Proper-time Hamiltonian for the self-gravitating shell

We now apply the time-reparametrization formalism of Sec. IV A to the Hamiltonian theory (3.16). Our goal is to obtain a Hamiltonian that evolves the shell with respect to its proper time. We follow the route of Sec. IV A 2, specifying the reparametrization in terms of the new velocity. M_- will be regarded as a prescribed constant throughout.

We first need the Hamiltonian $h=M_++M_-$ (3.10) as a function of the old velocity $\dot{\mathfrak{r}}$. Using the implicit relations (3.8b) and (3.9) to evaluate $\partial M_+/\partial p$, we find that Hamilton's equation $\dot{\mathfrak{r}}=\partial h/\partial p$ takes the form

$$\dot{\mathfrak{r}}=\frac{\mathfrak{p}}{\sqrt{\mathfrak{p}^2+m^2}}-\sqrt{\frac{2M_+}{\mathfrak{r}}}, \tag{4.15}$$

where $\mathfrak{p}$ is still implicitly given by Eq. (3.9). Solving Eq. (4.15) for $\mathfrak{p}$ and substituting in Eq. (3.9) yields

$$\frac{M_+-M_-}{m}-\frac{m}{2\mathfrak{r}}=\frac{1-(\dot{\mathfrak{r}}+\sqrt{2M_+/\mathfrak{r}})\sqrt{2M_+/\mathfrak{r}}}{\sqrt{1-(\dot{\mathfrak{r}}+\sqrt{2M_+/\mathfrak{r}})^2}}. \tag{4.16}$$

Equation (4.16) determines M_+, and hence h, as a function of $\mathfrak{r}$ and $\dot{\mathfrak{r}}$.

Let τ denote the proper time of the shell. As the parameter time t coincides with the spatially flat time t_1 in the metric (3.4b) right of the shell, we have

$$d\tau^2=dt^2-\left(d\mathfrak{r}+\sqrt{\frac{2M_+}{\mathfrak{r}}}\,dt\right)^2. \tag{4.17}$$

This can be solved for $dt/d\tau$ as

$$\frac{dt}{d\tau}=\frac{\sqrt{2M_+/\mathfrak{r}}\ (d\mathfrak{r}/d\tau)+\tilde{\epsilon}_+\sqrt{(d\mathfrak{r}/d\tau)^2+1-2M_+/\mathfrak{r}}}{(1-2M_+/\mathfrak{r})}, \tag{4.18}$$

where the parameter $\tilde{\epsilon}_+=\pm 1$ labels the two solutions. Using Eq. (4.18) to express $\dot{\mathfrak{r}}$ in terms of $d\mathfrak{r}/d\tau$, we can put Eq. (4.16) in the form

$$\frac{M_+-M_-}{m}-\frac{m}{2\mathfrak{r}}=\tilde{\epsilon}_+\sqrt{(d\mathfrak{r}/d\tau)^2+1-2M_+/\mathfrak{r}}. \tag{4.19}$$

As M_+ is a constant of motion, the shell motion is completely determined by Eq. (4.19). Comparing Eq. (4.19) to Eq. (2.12) shows that our reduced Hamiltonian theory has correctly reproduced the shell motion that arises from Israel's junction condition formalism, with the parameter $\tilde{\epsilon}_+$ coinciding with the parameter ϵ_+ in Eq. (2.12). Equation (4.19) results from squaring Eq. (2.12) once, in a way that eliminates the parameter ϵ_-; however, as $\epsilon_-=1$ by our global assumptions, the full information in Eq. (2.12) is contained in Eq. (4.19).

Solving Eq. (4.19) for M_+ yields

$$\frac{M_+-M_-}{m}+\frac{m}{2\mathfrak{r}}=\sqrt{(d\mathfrak{r}/d\tau)^2+1-2M_-/\mathfrak{r}}. \tag{4.20}$$

As $M_+>M_-$, only the positive sign for the square root in Eq. (4.20) can occur; in terms of Eq. (2.12), this sign is equal to ϵ_-. From Eq. (4.20), the energy function on the new velocity space reads

$$\tilde{H}(\mathfrak{r},V)=M_+(\mathfrak{r},V)+M_-=m\sqrt{V^2+1-2M_-/\mathfrak{r}}-\frac{m^2}{2\mathfrak{r}}+2M_-, \tag{4.21}$$

where we have written, in the notation of Sec. IV A, $V:=d\mathfrak{r}/d\tau$. As a particular solution to Eq. (4.10) we choose

$$P(\mathfrak{r},V)=m\ln(V+\sqrt{V^2+1-2M_-/\mathfrak{r}}). \tag{4.22}$$

Inverting this for V and substituting in Eq. (4.21) gives the new Hamiltonian

$$H(\mathfrak{r},P)=m\ \cosh(P/m)-\frac{m^2}{2\mathfrak{r}}+M_-\left[2-\left(\frac{m}{\mathfrak{r}}\right)\exp(-P/m)\right]. \tag{4.23}$$

C. Minkowski-like Hamiltonian for the self-gravitating shell

We now consider a time reparametrization that makes the shell Hamiltonian analogous to the Minkowski time point particle Hamiltonian (4.11), which is also the Minkowski time Hamiltonian for a free spherical, nongravitating dust shell in flat space. Starting from the shell proper-time Hamiltonian (4.23), we denote the new momentum by $\mathbf{p}$, and we run the formalism of Sec. IV A 1 backwards with the choice $N(\mathfrak{r},\mathbf{p})=m(\mathbf{p}^2+m^2)^{-1/2}$. As with the point particle example in Sec. IV A 4, we solve Eq. (4.3) by $\hat{P}(\mathfrak{r},\mathbf{p}) = m\sinh^{-1}(\mathbf{p}/m)$. Denoting the counterpart of h in Eq. (4.4) by $\mathbf{h}(\mathfrak{r},\mathbf{p})$, we obtain

$$\mathbf{h}(\mathfrak{r},\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2} - \frac{m^2}{2\mathfrak{r}} + M_-\left[2 - \frac{1}{\mathfrak{r}}\left(\sqrt{\mathbf{p}^2+m^2}-\mathbf{p}\right)\right]. \tag{4.24}$$

V. $\mathbb{RP}^3$ GEON WITH A SELF-GRAVITATING SHELL

In this section we adapt the formalism to a shell in a spacetime with the $\mathbb{RP}^3$ geon topology.

As mentioned in the Introduction, an asymptotically flat, spherically symmetric spacetime (M,g) with a single asymptopia can have spatial topology $\mathbb{RP}^3 \setminus \{$a point at infinity$\}$. We refer to an asymptotically flat spacetime with this topology, or to an asymptotically flat initial data set in such a spacetime, as an $\mathbb{RP}^3$ geon. The covering space of the spacetime then has the wormhole topology of the extended Schwarzschild geometry.

In vacuum, one can obtain a spherically symmetric $\mathbb{RP}^3$ geon Einstein spacetime as the quotient of Kruskal manifold under a freely and properly discontinuously acting involutive isometry [1]. Let $(\overline{M},\overline{g})$ be Kruskal manifold, and let $(\tilde{t},\tilde{x},\theta,\phi)$ be a chart in which $\tilde{t}$ and $\tilde{x}$ are the usual Kruskal time and space coordinates (denoted, respectively, by v and u in Ref. [33]). The isometry in question is then

$$I:(\tilde{t},\tilde{x},\theta,\phi)\mapsto(\tilde{t},-\tilde{x},\pi-\theta,\phi+\pi). \tag{5.1}$$

As I commutes with rotations, the quotient spacetime $(M,g) := (\overline{M},\overline{g})/I$ is spherically symmetric. In $(\overline{M},\overline{g})$, the constant $\tilde{t}$ hypersurfaces that do not hit a singularity have topology $S^2\times\mathbb{R}$, with two asymptotically flat infinities, and they have at $\tilde{x}=0$ a wormhole throat at which the radius of the S^2 reaches its minimum value. In (M,g), the corresponding constant $\tilde{t}$ hypersurfaces have topology $\mathbb{RP}^3 \setminus \{$a point at infinity$\}$, and the throat has become a ''minimum radius'' two-surface with topology $\mathbb{RP}^2$. Away from the throat history, (M,g) is indistinguishable from half (say, $\tilde{x}>0$) of $(\overline{M},\overline{g})$. The Penrose diagram can be found in Ref. [1]. Note that the throat history in $(\overline{M},\overline{g})$ is only defined with respect to a given foliation, while the throat history in (M,g) has a coordinate invariant meaning as the trajectory of the ''minimum radius'' $\mathbb{RP}^2$. The reason for this difference is that I does not commute with the Killing time translations on $(\overline{M},\overline{g})$: these Killing time translations do not descend into globally defined isometries of (M,g).

Consider now a spherically symmetric spacetime that has the $\mathbb{RP}^3$ geon topology and solves Einstein's equations with a spherical dust shell. Away from the shell, Birkhoff's theorem still guarantees that the spacetime is locally isometric to Kruskal manifold. We assume that the spacetime right of the shell is as in Sec. III: this part of the spacetime is part of Kruskal geometry, containing the right-hand-side Kruskal infinity, and the shell trajectory intersects the right-hand-side exterior region in this Kruskal geometry. The spacetime left of the shell is assumed to be part of the vacuum $\mathbb{RP}^3$ geon spacetime described above, and to contain the throat history.

If the shell passes through the throat, it needs to cross itself there. We assume that such a crossing does not happen.

A Cauchy surface in this spacetime has only one infinity, in the part right of the shell, whereas the part left of the shell is compact. We can therefore unambiguously regard the left-hand side of the shell as the interior and the right-hand side as the exterior.

It is easy to adapt the Hamiltonian formalism of Sec. II to these $\mathbb{RP}^3$ boundary conditions. We take N, Λ, R, π_Λ, and π_R to be even in r and N^r odd in r, with the consequence that $M_+ = M_-$ in the falloff (2.16). We assume $\mathfrak{r}>0$, add to the system a second shell at $r=-\mathfrak{r}$, and finally take the quotient of the spacetime under the isometry $(t,r,\theta,\phi)\mapsto(t,-r,\pi-\theta,\phi+\pi)$. The resulting Hamiltonian theory is clearly consistent in the same sense as the Kruskal-type theory of Sec. II. The action can be written as

$$S = S_\Sigma + S_{\partial\Sigma}, \tag{5.2}$$

where S_Σ is given by Eq. (2.5), with the r integration extending from $r=0$ to $r=\infty$, and

$$S_{\partial\Sigma} = -\int dt\; M_+ . \tag{5.3}$$

When the equations of motion hold, we recover the above $\mathbb{RP}^3$ Einstein spacetimes with a dust shell. The throat is located at $r=0$.

The Hamiltonian reduction proceeds in close analogy with that in Sec. III. To choose the gauge, we introduce the analogue of the chart C_1 (3.4), with $M_->0$ now denoting the mass in the interior. The range of r_1 is bounded below by the t_1-dependent throat radius, and it is the throat radius that emerges as the parameter ρ. The transition region near the throat is handled as in Appendix A, but because now $r>0$ in our action, the contribution to the Liouville form is only half of that found in Appendix A. The transition region near the shell is handled exactly as in Appendix B. The reduced action is given by Eqs. (3.8)–(3.11), with the exceptions that the right-hand side in the counterpart of Eq. (3.8a) contains the factor $\frac{1}{2}$, and Eq. (3.10) is replaced by

$$h := M_+ . \tag{5.4}$$

From Sec. III it is clear that the reduced theory reproduces the correct equations of motion. In the classical solutions, $\rho(t)$ is the value of R at the throat in a foliation defined as with the chart (3.14).

A canonical transformation that replaces the pair (ρ,p_ρ) by (M_-,P_-) leads to the action (3.13), with Eq. (3.10) replaced by (5.4). Dropping the term $P_-\dot{M}_-$ gives a theory in which the interior mass M_- is regarded as a prescribed positive constant. The time reparametrizations of Sec. IV clearly carry through without change: the counterparts of the Hamiltonians (4.23) and (4.24) differ only in that the (constant) additive term $2M_-$ is replaced by M_-.

VI. SELF-GRAVITATING SHELL WITH $\mathbb{R}^3$ SPATIAL TOPOLOGY

In this section we consider the spatial topology $\mathbb{R}^3$.

We start directly from the action principle. In the bulk action (2.5), we take $0<r<\infty$, with the falloff (2.16) at $r\rightarrow\infty$. The total action is given by Eqs. (5.2) and (5.3). At $r\rightarrow 0$, we introduce the falloff

$$\Lambda(t,r)=\Lambda_0+O(r^2), \tag{6.1a}$$

$$R(t,r)=R_1 r+O(r^3), \tag{6.1b}$$

$$\pi_\Lambda(t,r)=\pi_{\Lambda_2}r^2+O(r^4), \tag{6.1c}$$

$$\pi_R(t,r)=\pi_{R_1}r+O(r^3), \tag{6.1d}$$

$$N(t,r)=N_0+O(r^2), \tag{6.1e}$$

$$N^r(t,r)=N_1^r r+O(r^3), \tag{6.1f}$$

where $\Lambda_0>0$, $R_1>0$, π_{Λ_2}, π_{R_1}, $N_0>0$, and N_1^r are functions of t only. It is straightforward to verify that this falloff is consistent with the constraints and preserved by the time evolution, and no additional boundary terms in the action are needed at $r=0$. From Eq. (3.2) we see that in the classical solutions, the mass left of the shell must vanish, and $r=0$ is just the coordinate singularity at the center of hyperspherical coordinates in flat space. The classical solutions therefore describe a self-gravitating shell with a flat interior. The spatial topology is $\mathbb{R}^3$.

The reduction proceeds as above, using the analogue of the chart C_1 (3.4) with $M_-=0$ and $r_1>0$. In the region $r_1<\mathfrak{r}-l$, the initial data hypersurface Σ_0 extends smoothly to $r_1=0$, and there is no counterpart of the parameter ρ of the Kruskal and $\mathbb{RP}^3$-geon topologies. The only contribution to the Liouville form comes from the shell transition region, which is handled exactly as in Appendix B but with $M_-=0$. The reduced action is given by Eqs. (3.16) and (5.4), where M_+ is obtained from Eqs. (3.8b) and (3.9) with $M_-=0$. It is again clear from Sec. III that this reduced theory reproduces the correct dynamics. As expected, the reduced phase space is two dimensional.

The time reparametrizations of Sec. IV carry through without change. The counterparts of the Hamiltonians (4.23) and Eq. (4.24) are obtained from these formulas by simply setting $M_-=0$. In particular, (4.24) reduces to the Hamiltonian used in Refs. [39,41].

VII. REMARKS ON QUANTIZATION

In this section we discuss the prospects for quantizing the reduced theories. We first review the pure vacuum case, and then turn to the coupled system.

A. Mass spectrum of spherically symmetric vacuum wormholes and $\mathbb{RP}^3$ geons

In Secs. II–VI we considered the dynamics of a shell coupled to spacetime geometry. However, the methods immediately adapt to spherically symmetric vacuum gravity by simply omitting the shell.

With the Kruskal topology, and no asymptotic masses fixed, the constraints imply $M_+=M_-:=M$. With the gauge choice of Sec. III, without the shell, the reduced action reads

$$S=\int dt\ (p_\rho\dot{\rho}-h), \tag{7.1}$$

where $h=2M$, and M is obtained from Eq. (3.8a) with $M_-=M$. Geometrically, $\rho(t)$ is the value of R at the sharp ridge in the foliation described in Sec. III B, without the shell; this ridge evolves in the black hole interior along a history of constant Killing time. At $t\rightarrow-\infty$, we have $\rho(t)\rightarrow 2M$ as the ridge approaches the bifurcation two-sphere, but in the future the gauge breaks down at a finite value of t as $\rho(t)\rightarrow 0$. With the $\mathbb{RP}^3$-geon topology, the only differences are that $h=M$, and the right-hand side of Eq. (3.8a) contains an additional factor $\frac{1}{2}$. The reduced phase space is two-dimensional in each case. These results agree with those obtained by Kuchař's reduction method [6,7] under a falloff that is qualitatively similar but makes the constant t hypersurfaces asymptotic to hypersurfaces of constant Minkowski time.

If one chooses to fix the mass at one infinity with the Kruskal topology, the reduced theory has no degrees of freedom. The same holds if one chooses to fix the mass at the infinity or at the throat with the $\mathbb{RP}^3$ geon topology. With the $\mathbb{R}^3$ topology, the reduced theory is always void.

Quantizing the reduced theories with a zero-dimensional reduced phase space is of course trivial: the mass M is a prescribed c number. Quantizing the theories with a two-dimensional reduced phases space offers, however, several options.

One option is to perform first a canonical transformation to the pair (M,P_M) as in Secs. III and V. One can then take quantum states to be described by functions $\Psi(M)$ of the positive-valued configuration variable M, adopt the inner product $\langle\Psi_1|\Psi_2\rangle=\int_0^\infty dM\ \overline{\Psi_1(M)}\Psi_2(M)$ (or a similar inner product with some M-dependent weight factor), and promote M into the quantum operator $\hat{M}$ that acts in the Schrödinger picture as [4–6]

$$\hat{M}\Psi(M)=M\Psi(M). \tag{7.2}$$

The spectrum of $\hat{M}$, and thus also that of the Hamiltonian operator $\hat{h}$, is continuous and consists of the positive real axis.

Another option is to take quantum states to be described by functions $\psi(\rho)$ of the positive-valued "throat radius" ρ, adopt the inner product $\langle\psi_1|\psi_2\rangle=\int_0^\infty\mu(\rho)d\rho\ \overline{\psi_1(\rho)}\psi_2(\rho)$ where $\mu(\rho)$ is some weight factor, and try to promote the function $M(\rho,p_\rho)$ into an operator on this Hilbert space. As our $M(\rho,p_\rho)$ is known only implicitly, we have not tried to pursue this quantization, but there seems no obvious reason to expect that the spectral properties of the resulting Hamiltonian operator would agree with those of the operator $\hat{M}$ in Eq. (7.2).

Indeed, quantization of spherically symmetric vacuum gravity was discussed in Ref. [12] in terms of a related ''wormhole throat'' phase space (a,p_a), on which the Schwarzschild mass is given by

$$M(a,p_a)=\frac{1}{2}\left(\frac{p_a^2}{a}+a\right). \tag{7.3}$$

The configuration variable a has an interpretation as the radius of the wormhole throat, much as our ρ, but with a time parameter that is now identified with the proper time of the throat history.[10] If the Hilbert space is chosen as above with the configuration variable ρ, with reasonable choices for the weight factor μ, the function $M(a,p_a)$ can be promoted into a self-adjoint operator whose spectrum is bounded below and purely discrete [12].

We regard as artificial the continuous mass spectrum arising from the quantization (7.2), because one can similarly obtain a continuous spectrum for *any* dynamical system whose Hamiltonian is not explicitly time dependent. For any function H with nonvanishing gradient on the phase space, one can find a local canonical chart of the form $(H,q_2,\ldots,q_n,p_H,p_2,\ldots,p_n)$, in which H is one of the canonical coordinates. If the range of H in this chart is $\mathbb{R}_+$, one can adopt a Schrödinger representation with Hilbert space $L_2(\mathbb{R}_+)\otimes\mathsf{H}$, with H a Hilbert space for the remaining q's. The Hamiltonian operator $\hat{H}$ can then be taken to act as a multiplication operator,

$$\hat{H}\psi(H,q_2,\ldots,q_n)=H\psi(H,q_2,\ldots,q_n), \tag{7.4}$$

and its spectrum is $\mathbb{R}_+$.

Ambiguities in canonical quantization are, of course, well recognized [61–63]. One specific issue not addressed above is in the global properties of the canonical transformations. For example, the canonical transformation that takes the phase space (a,p_a) to Kuchař's reduced phase space [6] is not onto: the classical dynamics in Kuchař's reduced phase space is complete, but the classical dynamics in the phase space (a,p_a) is not [12]. One's attitude to such classical incompleteness in view of quantization may depend on what one sees as the role of singularities in quantum gravity [12,64–70].

B. Quantization of shell coupled to geometry

We now turn to the coupled system. We restrict consideration to the proper-time Hamiltonian (4.23).

When $M_-=0$, the shell encloses a flat interior with trivial topology, and the Hamiltonian (4.23) takes the form corresponding to a relativistic particle in a Coulomb potential,

$$H(\mathfrak{r},P)=m\cosh(P/m)-\frac{m^2}{2\mathfrak{r}}, \tag{7.5}$$

discussed by Hájíček [38]. One can adopt a Schrödinger representation corresponding to configuration-space variable $\mathfrak{r}\in\mathbb{R}_+$ and Hilbert space

$$\mathsf{H}:=L_2(\mathbb{R}_+,r^\alpha d\mathfrak{r}), \tag{7.6}$$

where α is a parameter. With the factor ordering

$$\widehat{[\cosh(P/m)}=\lim_{N\to\infty}\mathfrak{r}^{-\alpha/2}\sum_{n=0}^{N}\frac{(-1)^n}{(2n)!}\Delta^n\mathfrak{r}^{\alpha/2}, \tag{7.7}$$

where $\Delta=\partial_{\mathfrak{r}}^2$, H becomes a self-adjoint operator $\hat{H}$ with domain [71]

$$D(\hat{H})=\{f|f^{(2n)}(0)=0,\ f^{(n)}\in L_2,\quad \text{all}\ n\}. \tag{7.8}$$

[Hájíček takes $\alpha=-1$, but notes the unitary equivalence of $(\mathsf{H},\hat{H})$ for a different choice of α.] For $m<1.9$, $\hat{H}$ is bounded below, and its spectrum, like that of the nonrelativistic Coulomb problem, has discrete and continuous parts.

When $M_->0$, one expects that the Hamiltonian (4.23) can be made into a self-adjoint operator in an analogous manner, and one expects the spectrum then to be bounded below and partly discrete for small values of m. However, there appears to be no reason to expect that the term proportional to M_- would allow the spectrum to have a lower bound for large values of m. Oharu and Winters-Hilt [71] are currently examining a self-adjoint extension of H on $L_2(\mathbb{R}_+,d\mathfrak{r})$, with factor ordering corresponding to the choice (7.7) with $\alpha=0$:

$$\hat{H}=m\widehat{[\cosh(P/m)]}-\frac{m^2}{2\mathfrak{r}}-M_-m\mathfrak{r}^{-1/2}\widehat{[\exp(-P/m)]}\mathfrak{r}^{-1/2}+2M_-. \tag{7.9}$$

Finally, recall that our time-reparametrization derivation of the proper-time Hamiltonian (4.23) assumed M_- to be a prescribed, time-independent constant. With $\mathbb{R}^3$ spatial topology this assumption is automatically satisfied. With the Kruskal and $\mathbb{RP}^3$-geon topologies, on the other hand, one could ask whether it is still possible to carry out an analogous time reparametrization when M_- is a dynamical variable and the phase space is four dimensional. If the answer is affirmative, one could presumably raise anew the issues regarding the spectrum of $\hat{M}_-$ that were addressed in the context of the vacuum theory in Sec. VII A. If, after the reparametrization, the dynamics of M_- still decouples from the dynamics of the shell as in Secs. III and V, one could effectively separate variables by first considering the eigenvalue equation for $\hat{M}_-$,

$$\hat{M}_-\psi=M_-\psi. \tag{7.10}$$

[10] The Hamiltonian $M(a,p_a)$ (7.3) describing the proper-time evolution of the throat was previously considered by Friedman, Redmount and Winters-Hilt [59,60] without a derivation by reduction from spherically symmetric vacuum gravity. In Ref. [12], this Hamiltonian was derived from Kuchař's reduced Hamiltonian theory [6] by a canonical transformation. A similar derivation could clearly be given from the canonical pair (M,P_M) of the present paper, despite the technical differences in our falloff and that of Ref. [6].

For each eigenspace of $\hat{M}_-$, the shell Hamiltonian would then have the form (4.23) with a c-number M_-, and the character of the total spectrum would depend on the spectrum of $\hat{M}_-$.

VIII. SUMMARY AND DISCUSSION

In this paper we have considered the Hamiltonian dynamics of spherically symmetric spacetimes that contain an idealized, infinitesimally thin massive dust shell. We considered the Kruskal-like spatial topology $S^2\times\mathbb{R}$, the $\mathbb{RP}^3$-geon spatial topology $\mathbb{RP}^3\setminus$ {a point at infinity}, and the Euclidean spatial topology $\mathbb{R}^3$. The variational equations that arose from the unreduced Hamiltonian action were not strictly consistent in a distributional sense, but we were able to localize the ambiguity into the single equation that arises by varying the action with respect to the shell position. When the ambiguous contribution to this equation was interpreted as the average of its values on the two sides of the shell, we correctly reproduced the content of Israel's junction condition formalism.

We performed a Hamiltonian reduction by adopting a gauge with piecewise flat spatial sections, and passing to the limit in which the interpolating transition regions became vanishingly narrow. The constraints could then be explicitly solved. For the Kruskal and $\mathbb{RP}^3$ topologies the reduced phase space was four dimensional, with one canonical pair closely associated with the shell motion and the other pair with the dynamics of the geometry. In the limit where the shell is not present, this correctly reproduced previous results for spherically symmetric vacuum geometries. Retaining the shell but prescribing by hand one asymptotic mass for the Kruskal topology, and the interior mass for the $\mathbb{RP}^3$ topology, we recovered theories whose reduced phase space was two dimensional, with just the canonical pair associated with the shell motion surviving. For the $\mathbb{R}^3$ topology, the interior mass necessarily vanishes, and we only obtained a two-dimensional phase space, with the single canonical pair describing the shell motion.

For each of the three spatial topologies, we time-reparametrized the dynamics in the two-dimensional phase space that describes the shell motion with fixed interior mass. With one choice for the reparametrization, we recovered a previously known Hamiltonian that evolves the shell with respect to its proper time. With another choice, we recovered a Hamiltonian analogous to the square-root Hamiltonian of a spherical test shell in Minkowski space. Finally, we briefly discussed the spectra that would be expected to emerge in different approaches of canonically quantizing the theories.

Our results provide a robust description of the reduced Hamiltonian dynamics of a spherically symmetric dust shell coupled to gravity, in the region of the reduced phase space that is covered by our piecewise spatially flat gauge. While this gauge is not global, one can argue that this gauge and its time-inverted counterpart cover the region of the reduced phase space that is of interest to an observer who scrutinizes the shell motion from one asymptotically flat infinity. What remains open, however, is the global structure of the reduced phase space. One would also like to describe the reduced phase space in a way that is more geometrical and less tied to a particular gauge. One possible avenue for this, currently under investigation by Hájíček and Kijowski [72,73], might be to generalize to the massive shell the canonical transformations that Kuchař introduced to simplify the vacuum theory [6]. Work on the analogous problem with a null-dust shell is in progress [74].

More ambitiously, one would like to consider systems with matter that is more interesting than a dust shell. The canonical formulation of Einstein gravity coupled to a *continuous* distribution of massive or null dust has been discussed, respectively, in Refs. [75,76]. For the canonical formulation in the presence of other types of fluids, see, for example, Ref. [77] and the references therein. A discussion of the difficulties involved with spherically symmetric gravity coupled to a scalar field is given in Refs. [78,79]. A discussion in the context of a dilatonic black hole can be found in Ref. [80].

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APPENDIX A: RIDGE TRANSITION REGION

In this appendix we specify the gauge in the ridge transition region $|r|\leq(1+\gamma)\rho$, and evaluate the contribution from this region to the integral on the right-hand side of Eq. (3.3) in the limit $\gamma\to 0$.

1. Gauge choice

To specify the gauge in the region $|r|\leq(1+\gamma)\rho$, we consider the classical spacetime of Sec. III B, and the spacelike hypersurface $\tilde{\Sigma}_0$ in this spacetime. The part $|r|\leq(1+\gamma)\rho$ of $\tilde{\Sigma}_0$ lies in the black-hole region of the Kruskal spacetime left of the shell.

Let $h:\mathbb{R}\to\mathbb{R}$ be a smooth function such that

$$h(x)=\begin{cases}0, & x\leq 0\\ x-\frac{1}{2}, & x\geq 1,\end{cases} \tag{A1}$$

and $d^2h/dx^2>0$ for $0<x<1$. We write $h^{(n)}(x) := d^n h(x)/dx^n$.

For $|r|\leq(1+\gamma)\rho$, we seek a gauge in the form

$$\Lambda(r)=(1-\Lambda_0)h^{(1)}\left(\frac{|r|-\rho}{\gamma\rho}\right)+\Lambda_0, \tag{A2a}$$

$$R(r)=\left[\gamma h\left(\frac{|r|-\rho}{\gamma\rho}\right)+1+\tfrac{1}{2}\gamma\right]\rho, \tag{A2b}$$

where Λ_0 is a positive parameter. In the subregion $|r|\leq\rho$, the radius of the two-sphere is constant on $\tilde{\Sigma}_0$, $R=(1+\frac{1}{2}\gamma)\rho$, and the proper distance on $\tilde{\Sigma}_0$ is $\Lambda_0 dr$. The subregions $\rho \leqslant |r| \leqslant (1+\gamma)\rho$ interpolate smoothly between this constant radius gauge and the spatially flat gauge (3.6). Note that $\Lambda(r) > 0$, and $(1+\frac{1}{2}\gamma)\rho \leqslant R \leqslant (1+\gamma)\rho$.

Recall from Sec. III B that $(1+\gamma)\rho < 2M_-$. Equations (3.2) thus yield a real-valued solution for π_Λ and π_R for all of $|r| \leqslant (1+\gamma)\rho$. The gauge (A2) therefore specifies a spacelike hypersurface in an interior Kruskal geometry with mass M_-, with the ends at $R = (1+\gamma)\rho$. What remains is to choose the parameter Λ_0 in Eq. (A2a) so that this hypersurface precisely fits between the points $|r| = (1+\gamma)\rho$.

In the curvature coordinates (T,R) in the black-hole interior, the metric reads

$$ds^2 = -\left(\frac{2M_-}{R} - 1\right)^{-1} dR^2 + \left(\frac{2M_-}{R} - 1\right) dT^2 + R^2 d\Omega^2, \tag{A3}$$

where $0 < R < 2M_-$, R decreases to the future, and we take T to increase to the right. The transformation from Eq. (A3) to the chart C_1 of Sec. III B reads

$$T = t_1 - 2\sqrt{2M_- r_1} - 2M_- \ln\left(\frac{\sqrt{2M_-} - \sqrt{r_1}}{\sqrt{2M_-} + \sqrt{r_1}}\right) + \text{ const}, \tag{A4a}$$

$$R = r_1. \tag{A4b}$$

As our (prospective) deformation (A2) of $\hat{\Sigma}_0$ to $\tilde{\Sigma}_0$ is symmetric around $r=0$, the value of T at $r=0$ on $\tilde{\Sigma}_0$ is the same as the value of T at the unsmoothed ridge on $\hat{\Sigma}_0$. On $\tilde{\Sigma}_0$, we thus have

$$T_{r=(1+\gamma)\rho} - T_{r=0} = 2\sqrt{2M_-\rho} - 2\sqrt{2M_-(1+\gamma)\rho} + 2M_- \ln\left[\frac{(\sqrt{2M_-} - \sqrt{\rho})(\sqrt{2M_-} + \sqrt{(1+\gamma)\rho})}{(\sqrt{2M_-} + \sqrt{\rho})(\sqrt{2M_-} - \sqrt{(1+\gamma)\rho})}\right]. \tag{A5}$$

On the other hand, from Eq. (80) of Ref. [6] we have

$$T' = \frac{\Lambda \pi_\Lambda}{2M_- - R}. \tag{A6}$$

Integrating Eq. (A6) from $r=0$ to $r=(1+\gamma)\rho$, with π_Λ given by Eq. (3.2a), and equating the result to Eq. (A5), gives a relation that implicitly determines Λ_0 in terms of M_-, ρ, and γ. By the symmetry of $\tilde{\Sigma}_0$ around $r=0$, the relation obtained by similarly comparing $T_{r=-(1+\gamma)\rho}$ to $T_{r=0}$, using the chart C_2, contains exactly the same information. This completes the gauge choice.

We shall below be interested in the limit of small γ. In this limit, the relation determining Λ_0 admits a power series expansion in γ. The result is

$$\Lambda_0 = \frac{\gamma}{2\sqrt{1-\rho/(2M)}} + O(\gamma^2), \tag{A7}$$

where O stands for a γ-dependent quantity that is bounded by a constant times its argument.

2. Liouville form

We now evaluate the contribution to the integral in the Liouville form (3.3) from $|r| \leqslant (1+\gamma)\rho$, in the limit $\gamma \to 0$.

As we have noted, the gauge (A2) for $|r| \leqslant (1+\gamma)\rho$ joins smoothly to the spatially flat gauge outside this interval, and the expressions given in (A2) are in fact valid for all of $-\infty < r < \mathfrak{r} - l$. The differentials $\boldsymbol{\delta}\Lambda(r)$ and $\boldsymbol{\delta}R(r)$ therefore contain no δ functions in r at $|r| = (1+\gamma)\rho$, and it is sufficient to consider the contributions from $|r| < \rho$ and $\rho < |r| < (1+\gamma)\rho$.

For $|r| < \rho$, we have $h = h^{(1)} = 0$. Equations (A2) and (A7) yield $\boldsymbol{\delta}\Lambda = O(\gamma)$ and $\boldsymbol{\delta}R = O(1)$, and Eqs. (3.2) yield $\pi_\Lambda = O(1)$ and $\pi_R = O(\gamma)$. The contribution to Eq. (3.3) is therefore $O(\gamma)$.

Suppose then $\rho < r < (1+\gamma)\rho$. We now obtain

$$\boldsymbol{\delta}\Lambda = -\frac{h^{(2)}}{\gamma\rho}\,\boldsymbol{\delta}\rho + O(1), \tag{A8a}$$

$$\boldsymbol{\delta}R = (1 - h^{(1)})\,\boldsymbol{\delta}\rho + O(\gamma), \tag{A8b}$$

$$\pi_\Lambda = \rho\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R} + O(\gamma), \tag{A8c}$$

$$\pi_R = \frac{\rho(R'/\Lambda)'}{\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R}} + O(1), \tag{A8d}$$

where the argument of h and its derivatives is always $(r-\rho)/\gamma\rho$. Note that the first term in Eq. (A8d) is $O(\gamma^{-1})$.

For $\int dr\ \pi_\Lambda \boldsymbol{\delta}\Lambda$, changing the integration variable from r to $x := (r-\rho)/\gamma\rho$ gives

$$\int_\rho^{(1+\gamma)\rho} dr\ \pi_\Lambda \boldsymbol{\delta}\Lambda = -\rho\,\boldsymbol{\delta}\rho \int_0^1 dx\ h^{(2)}(x)\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R} + O(\gamma)$$

$$= -\sqrt{2M_-\rho}\ \boldsymbol{\delta}\rho \int_0^1 dx\ h^{(2)}(x) + o(1)$$

$$= -\sqrt{2M_-\rho}\ \boldsymbol{\delta}\rho + o(1), \tag{A9}$$

where $o(1)$ stands for a γ-dependent quantity that goes to zero as $\gamma \to 0$. We have used the fact that $\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R} \to \sqrt{2M_-/\rho}$ pointwise in x as $\gamma \to 0$, and taken the limit under the integral by dominated convergence.

For $\int dr\ \pi_R \boldsymbol{\delta} R$, the assumption $h^{(2)} > 0$ allows us to change the integration variable from r to $u := R'/\Lambda$. We obtain

$$\int_\rho^{(1+\gamma)\rho} dr\ \pi_R \boldsymbol{\delta} R = \rho\ \boldsymbol{\delta}\rho \int_\rho^{(1+\gamma)\rho} \frac{dr\ (R'/\Lambda)'(1-h^{(1)})}{\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R}} + O(\gamma)$$

$$= \rho\ \boldsymbol{\delta}\rho \int_0^1 \frac{du(1-h^{(1)})}{\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R}} + O(\gamma)$$

$$= \rho\ \boldsymbol{\delta}\rho \int_0^1 \frac{du}{\sqrt{u^2 - 1 + 2M_-/\rho}} + o(1)$$

$$= \frac{1}{2} \ln\left(\frac{\sqrt{2M_-} + \sqrt{\rho}}{\sqrt{2M_-} - \sqrt{\rho}} \right) \rho\ \boldsymbol{\delta}\rho + o(1). \tag{A10}$$

We have used the facts that $\sqrt{(R'/\Lambda)^2 - 1 + 2M_-/R} \to \sqrt{u^2 - 1 + 2M_-/\rho}$ and $h^{(1)} = u\Lambda_0[1 - u(1-\Lambda_0)]^{-1} \to 0$ pointwise in u as $\gamma \to 0$, and taken the limit under the integral by dominated convergence.

Adding the identical contributions from the region $-(1+\gamma)\rho < r < -\rho$, we find that the total contribution to the Liouville form (3.3) from the ridge transition region $|r| \leqslant (1+\gamma)\rho$ is

$$\int_{-(1+\gamma)\rho}^{(1+\gamma)\rho} dr\ (\pi_\Lambda \boldsymbol{\delta}\Lambda + \pi_R \boldsymbol{\delta} R) = \left[\rho\ \ln\left(\frac{\sqrt{2M_-} + \sqrt{\rho}}{\sqrt{2M_-} - \sqrt{\rho}} \right) - 2\sqrt{2M_-\rho} \right] \boldsymbol{\delta}\rho + o(1). \tag{A11}$$

APPENDIX B: SHELL TRANSITION REGION

In this appendix we specify the gauge in the shell transition region $\mathfrak{r} - l \leqslant r \leqslant \mathfrak{r}$, and evaluate the contribution from this region to the integral on the right-hand side of Eq. (3.3) in the limit $l \to 0$.

1. Gauge choice

To specify the gauge in the region $\mathfrak{r} - l \leqslant r \leqslant \mathfrak{r}$, we again consider the classical spacetime of Sec. III B, and the spacelike hypersurface $\tilde{\Sigma}_0$ in this spacetime. The part $\mathfrak{r} - l \leqslant r \leqslant \mathfrak{r}$ of $\tilde{\Sigma}_0$ lies in the Kruskal spacetime left of the shell.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x\ e^{-x^2/(1-x^2)}, & x \in (0,1), \\ 0, & x \notin (0,1). \end{cases} \tag{B1}$$

We write $f^{(n)}(x) := d^n f(x)/dx^n$. f is continuous everywhere, and smooth except at $x=0$, with $f^{(1)}(x) \to 1$ as $x \to 0_+$ and $f^{(1)}(x) \to 0$ as $x \to 0_-$. Note that $f^{(2)}(x) \to 0$ as $x \to 0_\pm$.

For $(1+\gamma)\rho < r < \infty$, we choose the gauge

$$\Lambda = 1, \tag{B2a}$$

$$R = r - \frac{l\sqrt{\mathfrak{p}^2 + m^2}}{\mathfrak{r}}\, f\left(\frac{\mathfrak{r} - r}{l} \right). \tag{B2b}$$

Outside the shell transition region $\mathfrak{r} - l \leqslant r \leqslant \mathfrak{r}$, this clearly agrees with the spatially flat gauge (3.6). To show that the gauge is admissible, we first note that for $\mathfrak{r} - l \leqslant r < \mathfrak{r}$, the constraints are solved by the gravitational momenta given by Eq. (3.2). At the shell, the Hamiltonian constraint (2.13a) is identically satisfied. The momentum constraint (2.13b) at the shell reads, using Eq. (3.2a),

$$\mathfrak{p} = \mathfrak{r}\sqrt{(R'_-)^2 - 1 + 2M_-/\mathfrak{r}} - \sqrt{2M_+\mathfrak{r}}, \tag{B3}$$

where Eq. (B2b) gives

$$R'_- = \frac{1 + \sqrt{\mathfrak{p}^2 + m^2}}{\mathfrak{r}}. \tag{B4}$$

The constraints can therefore be solved both at the shell and away from the shell, and the gauge is thus admissible. The gauge is smooth everywhere except at the shell, and at the shell it is consistent with the regularity assumptions of Sec. II.

2. Liouville form

We now evaluate the contribution to the integral in the Liouville form (3.3) from $\mathfrak{r} - l \leqslant r \leqslant \mathfrak{r}$, in the limit $l \to 0$.

As the gauge (B2) is smooth for $(1+\gamma)\rho < r < \infty$ except at the shell, the differentials $\boldsymbol{\delta}\Lambda(r)$ and $\boldsymbol{\delta} R(r)$ do not contain δ functions in r except possibly at $r = \mathfrak{r}$. Equation (B2a) shows that $\boldsymbol{\delta}\Lambda(r) = 0$ everywhere. It is therefore sufficient to consider separately $\pi_R \delta R$ for $\mathfrak{r}-l<r<\mathfrak{r}$, and the δ function contribution to $\pi_R \delta R$ at $r=\mathfrak{r}$.

For $\mathfrak{r}-l<r<\mathfrak{r}$, Eq. (B2b) gives

$$R' = 1 + \frac{\sqrt{\mathfrak{p}^2+m^2}}{\mathfrak{r}} f^{(1)}, \tag{B5a}$$

$$R'' = -\frac{\sqrt{\mathfrak{p}^2+m^2}}{l\mathfrak{r}} f^{(2)}, \tag{B5b}$$

$$\delta R = (1-R')\,\delta\mathfrak{r} + O(l), \tag{B5c}$$

where the argument of f and its derivatives is $(\mathfrak{r}-r)/l$. From Eq. (3.2b) we have

$$\pi_R = \frac{\mathfrak{r}R''}{\sqrt{R'^2-1+2M_-/\mathfrak{r}}} + O(1), \tag{B6}$$

where we have used the observations $R''=O(l^{-1})$ and $R=\mathfrak{r}+O(l)$. Note that the first term in Eq. (B6) is $O(l^{-1})$. We thus obtain, changing the integration variable from r to $v:=R'$,

$$\begin{aligned}
\int_{\mathfrak{r}-l}^{\mathfrak{r}} dr\ \pi_R \delta R &= \mathfrak{r}\,\delta\mathfrak{r} \int_{\mathfrak{r}-l}^{\mathfrak{r}} \frac{dr\ R''(1-R')}{\sqrt{R'^2-1+2M_-/\mathfrak{r}}} + O(l) \\
&= \mathfrak{r}\,\delta\mathfrak{r} \int_1^{R'_-} \frac{dv\ (1-v)}{\sqrt{v^2-1+2M_-/\mathfrak{r}}} + O(l) \\
&= \Bigg[\sqrt{2M_-\mathfrak{r}} - \sqrt{2M_+\mathfrak{r}} - \mathfrak{p} \\
&\quad + \mathfrak{r}\ln\left(\frac{\mathfrak{r}+\mathfrak{p}+\sqrt{\mathfrak{p}^2+m^2}+\sqrt{2M_+\mathfrak{r}}}{\mathfrak{r}+\sqrt{2M_-\mathfrak{r}}} \right) \Bigg] \delta\mathfrak{r} + O(l),
\end{aligned} \tag{B7}$$

where we have used Eq. (B4) for R'_-.

What remains is the δ function in δR at $r=\mathfrak{r}$. From Eq. (B2b) we have

$$\delta R = -\frac{l\sqrt{\mathfrak{p}^2+m^2}\ \delta\mathfrak{r}}{\mathfrak{r}}\ \delta(r-\mathfrak{r}) + \text{(nondistributional function of } r). \tag{B8}$$

From Eq. (3.2b), we have

$$\pi_R^+ = \tfrac{1}{2}\sqrt{2M_+/\mathfrak{r}}, \tag{B9a}$$

$$\pi_R^- = \frac{(\mathfrak{p}+\sqrt{2M_+\mathfrak{r}})^2 - M_-\mathfrak{r}}{\mathfrak{r}(\mathfrak{p}+\sqrt{2M_+\mathfrak{r}})}, \tag{B9b}$$

where we have used Eq. (B4) and the fact that $R''_-=0$. As π_R is not continuous at $r=\mathfrak{r}$, the product $\pi_R \delta R$ is not defined as a distribution, and the contribution to the Liouville form is ambiguous. However, as $\pi_R^{\pm}$ are both of order 1, and the δ function in δR (B8) is $O(l)$, we argue that the ambiguous contribution can be taken to vanish in the limit $l\to 0$. It is seen in the main text that this leads to a reduced Hamiltonian system that reproduces the correct dynamics.

The ambiguity in $\pi_R \delta R$ appears to have the same origin as the ambiguity of the equation of motion (2.8f) in the unreduced formalism: both involve varying the action with respect to $\mathfrak{r}$. Note that if the function f had been chosen so that $f^{(2)}(x)\not\to 0$ as $x\to 0_+$, R''_- and π_R^- would be nonvanishing and proportional to l^{-1}, and the above argument for the vanishing of the ambiguity in the limit $l\to 0$ would not apply.

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