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Hamiltonian thermodynamics of the Reissner–Nordström–anti-de Sitter black hole

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We consider the Hamiltonian dynamics and thermodynamics of spherically symmetric Einstein-Maxwell spacetimes with a negative cosmological constant. We impose boundary conditions that enforce every classical solution to be an exterior region of a Reissner–Nordström–anti-de Sitter black hole with a nondegenerate Killing horizon, with the spacelike hypersurfaces extending from the horizon bifurcation two-sphere to the asymptotically anti–de Sitter infinity. The constraints are simplified by a canonical transformation, which generalizes that given by Kuchař in the spherically symmetric vacuum Einstein theory, and the theory is reduced to its true dynamical degrees of freedom. After quantization, the grand partition function of a thermodynamical grand canonical ensemble is obtained by analytically continuing the Lorentzian time evolution operator to imaginary time and taking the trace. A similar analysis under slightly modified boundary conditions leads to the partition function of a thermodynamical canonical ensemble. The thermodynamics in each ensemble is analyzed, and the conditions that the (grand) partition function be dominated by a classical Euclidean black hole solution are found. When these conditions are satisfied, we recover, in particular, the Bekenstein-Hawking entropy. The limit of a vanishing cosmological constant is briefly discussed. [S0556-2821(96)01114-9]

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I. INTRODUCTION

Hawking's celebrated result of black hole radiation [1] and related developments [2–4] made it possible to consider thermodynamical equilibrium systems involving black holes in the manner first anticipated by Bekenstein [5,6]. At a semiclassical, ''phenomenological,'' level, a black hole thermodynamical equilibrium system can be introduced by simply immersing a radiating black hole in a heat bath such that the outgoing Hawking radiation balances the radiation that falls in from the bath [7–11]. At a deeper level, one aspires to construct a full thermodynamical equilibrium ensemble by starting from a quantum theory of gravity for black-hole-type geometries [12–17]. For reviews, see, for example, Refs. [8,16,18–20].

At the semiclassical level, the thermodynamical equilibrium configurations involving black holes tend to be unstable against thermal fluctuations [7,8]. The classic example is a Schwarzschild black hole in equilibrium with an asymptotically flat heat bath, in the approximation where the back reaction of the radiation on the geometry is neglected: the heat capacity in this instance is $-(8\pi T^2)^{-1}$, where T is the temperature measured at the infinity, and the fact that this heat capacity is negative indicates thermodynamical instability. While such instabilities are not unexpected in self-gravitating systems, they do pose an obstacle to constructing thermodynamical equilibrium ensembles from quantum gravity. This is because the existence of a thermodynamical ensemble implies the positivity of certain response functions associated with that ensemble [21]. For example, in the canonical ensemble the heat capacity is necessarily positive; consequently, a canonical ensemble of the usual kind does not appear to exist for Schwarzschild black holes in an asymptotically flat space [22].

To construct a thermodynamical ensemble appropriate for black hole geometries from a quantum theory of gravity, one thus needs to choose the boundary conditions for the ensemble in a judicious manner, motivated by the stability of the corresponding semiclassical equilibrium situations. One possibility is to replace an asymptotic infinity by a finite ''box'' at which the local temperature is then fixed [14,23–34]. The possibility on which we shall concentrate in this paper is to include a negative cosmological constant [18,35–39].

A negative cosmological constant makes classical black hole solutions asymptotically anti–de Sitter. We shall consider spherically symmetric spacetimes, and as the only matter field we include the spherically symmetric Maxwell field. All the relevant classical solutions then belong to the Reissner–Nordström–anti-de Sitter (RNAdS) family [40–43]. The temperature of the Hawking radiation is redshifted to zero at the asymptotically anti–de Sitter infinity, but from the rate at which the local Hawking temperature approaches zero one can extract a ''renormalized'' Hawking temperature, and this renormalized Hawking temperature can then be taken as one fixed quantity in the thermodynamical en-

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sembles [18,35,36,38]. We shall consider both the canonical ensemble, in which the electric charge is fixed, and the grand canonical ensemble, in which the electric potential difference between the event horizon and the infinity is fixed.

To quantize the theory and to build the equilibrium ensembles, we shall adapt the method introduced in Ref. [32] in the context of spherically symmetric vacuum geometries in the presence of a finite boundary. We shall first set up a classical Lorentzian Hamiltonian theory in which, on the classical solutions, the right end of the spacelike hypersurfaces is at the asymptotically anti–de Sitter infinity in an exterior region of a black hole spacetime, and the left end of the hypersurfaces is at the bifurcation two-sphere of a nondegenerate Killing horizon. We then canonically quantize this theory, and obtain the thermodynamical (grand) partition function by suitably continuing the Schrödinger picture time evolution operator to imaginary time and taking the trace. A crucial input is how to handle the analytic continuation at the bifurcation two-sphere. As in Ref. [32], we shall see that a continuation motivated by smoothness of Euclidean black hole geometries yields a (grand) partition function that is in agreement with that obtained via path integral methods.

To implement the method used in Ref. [32], one must be able to canonically quantize the Lorentzian theory in some practical fashion. In Ref. [32] this was achieved by using canonical variables that were first introduced by Kuchař under asymptotically flat, Kruskal-like boundary conditions [44]. In these variables the constraints of the vacuum theory become exceedingly simple, and the classical Hamiltonian theory can be explicitly reduced into an unconstrained Hamiltonian theory with just one canonical pair of degrees of freedom. We shall show that an analogous set of canonical variables exists for our system, and the classical Hamiltonian theory can again be explicitly reduced into an unconstrained Hamiltonian theory. Under boundary conditions tailored to the grand canonical ensemble, the reduced Lorentzian Hamiltonian theory has *two* canonical pairs of degrees of freedom;[1] under boundary conditions tailored to the canonical ensemble, the reduced Lorentzian Hamiltonian theory has just one pair of canonical degrees of freedom. Using these variables, it will be possible to construct a quantum theory and a (grand) partition function in close analogy with Ref. [32].

It will turn out that both the canonical ensemble and the grand canonical ensemble for our system are well defined. In particular, the appropriate thermodynamical response functions are positive. We shall also be able to give the conditions under which the (grand) partition function is dominated by a classical Euclidean solution. The grand canonical ensemble exhibits a transition from a region where a classical solution dominates to a region where no classical solution dominates, in close analogy with what happens with the spherically symmetric boxed vacuum canonical ensemble [24,25]. As in Refs. [24,25], one may see this as evidence for a phase transition between a black hole sector and a topologically different ''hot anti–de Sitter space'' sector. In the canonical ensemble we find evidence for this kind of a phase transition only in the special case when the charge vanishes. When the charge is nonvanishing, there occurs a different kind of phase transition in which the dominating contribution to the partition function shifts from one classical solution to another as the boundary data changes.

The rest of the paper is as follows. In Sec. II we set up a classical Hamiltonian theory under boundary conditions tailored to the grand canonical thermodynamical ensemble, paying special attention to the falloff conditions at the asymptotically anti–de Sitter infinity [46,47]. In particular, we choose to fix the values of the electric potential at the infinity and at the horizon, in a manner that will be made precise in terms of the falloff conditions. In Sec. III we perform the canonical transformation, and in Sec. IV the constraints are eliminated and the theory is reduced to its true dynamical degrees of freedom. In Sec. V we quantize the theory and obtain the grand partition function of the thermodynamical grand canonical ensemble. The thermodynamics in the grand canonical ensemble is analyzed in Sec. VI. Section VII outlines the corresponding classical, quantum-mechanical, and thermodynamical analyses under boundary conditions that fix the charge instead of the electric potential, and thus lead to the thermodynamical canonical ensemble.

The results are summarized and discussed in Sec. VIII. Some facts about the RNAdS solutions are collected in Appendix A. Finally, Appendix B outlines the classical Hamiltonian analysis and the quantization of the reduced Hamiltonian theory in the case where the cosmological constant vanishes and the asymptotically anti–de Sitter falloff conditions are replaced by asymptotically flat falloff conditions. With asymptotic flatness, neither the partition function nor the grand partition function turns out to be well defined, and we recover neither a canonical ensemble nor a grand canonical ensemble.

II. CANONICAL FORMULATION IN THE METRIC VARIABLES

In this section we present a Hamiltonian formulation of spherically symmetric electrovacuum spacetimes with a negative cosmological constant, with boundary conditions appropriate for the exterior of a RNAdS black hole with a nondegenerate horizon. Some relevant properties of the RNAdS metric are reviewed in Appendix A.

We consider the general spherically symmetric Arnowitt-Deser-Misner (ADM) metric

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 d\Omega^2, \qquad (2.1)$$

where $d\Omega^2$ is the metric on the unit two-sphere, and N, N^r, Λ, and R are functions of t and r only. The electromagnetic potential is taken to be described by the spherically symmetric one-form

$$A = \Gamma dr + \Phi dt, \qquad (2.2)$$

where Γ and Φ are functions of t and r only. The fact that this one-form is globally defined makes the electromagnetic bundle trivial, and will preclude the black hole from having a magnetic charge. The coordinate r takes the semi-infinite

[1]The conclusion of two canonical pairs of degrees of freedom for the spherically symmetric Einstein-Maxwell system with a cosmological constant was previously reached, under a different set of boundary conditions, in Ref. [45].

range $[0,\infty)$. Unless otherwise stated, we shall assume both the spatial metric and the spacetime metric to be nondegenerate. In particular, Λ, R, and N are taken to be positive. We shall work in natural units, $\hbar = c = G = 1$.

The action of the Einstein-Maxwell theory with a negative cosmological constant is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-{}^{(4)}g}\,({}^{(4)}R + 6\ell^{-2} - F^{\mu\nu}F_{\mu\nu}) + (\text{boundary terms}), \tag{2.3}$$

where ${}^{(4)}g$ is the determinant of the four-dimensional metric, ${}^{(4)}R$ is the four-dimensional Ricci scalar, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. The cosmological constant has been written as $-3\ell^{-2}$, where $\ell > 0$. Inserting the spherically symmetric fields (2.1) and (2.2) and integrating over the two-sphere we obtain, up to boundary terms, the action

$$\begin{aligned} S_\Sigma[\Lambda,R,\Gamma;N,N^r,\Phi] = \int dt\int_0^\infty dr\big(&-N^{-1}\{R[-\dot\Lambda + (\Lambda N^r)'](-\dot R + R'N^r) + \tfrac{1}{2}\Lambda(-\dot R + R'N^r)^2\} + \tfrac{1}{2}N^{-1}\Lambda^{-1}R^2(\dot\Gamma - \Phi')^2 \\ &+ N(\Lambda^{-2}RR'\Lambda' - \Lambda^{-1}RR'' - \tfrac{1}{2}\Lambda^{-1}R'^2 + \tfrac{1}{2}\Lambda + \tfrac{3}{2}\ell^{-2}\Lambda R^2)\big). \end{aligned} \tag{2.4}$$

The equations of motion derived from local variations of Eq. (2.4) are the full Einstein-Maxwell equations for the spherically symmetric fields (2.1) and (2.2). A generalized Birkhoff's theorem can be proven using the same techniques as in the case of a vanishing cosmological constant [48]: every classical solution is locally either a member of the extended RNAdS family (see Appendix A), or a spacetime that generalizes the Bertotti-Robinson solution to accommodate a negative cosmological constant [41–43,48]. We shall address the boundary conditions and boundary terms that are needed to make the variational principle globally well defined after passing to the Hamiltonian formulation.

The momenta conjugate to the configuration variables Λ, R, and Γ are

$$P_\Lambda = -N^{-1}R(\dot R - R'N^r), \tag{2.5a}$$

$$P_R = -N^{-1}\{\Lambda(\dot R - R'N^r) + R[\dot\Lambda - (\Lambda N^r)']\}, \tag{2.5b}$$

$$P_\Gamma = N^{-1}\Lambda^{-1}R^2(\dot\Gamma - \Phi'). \tag{2.5c}$$

A Legendre transformation gives the Hamiltonian action

$$S_\Sigma[\Lambda,R,\Gamma,P_\Lambda,P_R,P_\Gamma;N,N^r,\tilde\Phi] = \int dt\int_0^\infty dr(P_\Lambda\dot\Lambda + P_R\dot R + P_\Gamma\dot\Gamma - NH - N^rH_r - \tilde\Phi G), \tag{2.6}$$

where the super-Hamiltonian constraint H, the radial supermomentum constraint H_r, and the Gauss law constraint G are given by

$$\begin{aligned} H = {}&-R^{-1}P_RP_\Lambda + \tfrac{1}{2}R^{-2}\Lambda(P_\Lambda^2 + P_\Gamma^2) + \Lambda^{-1}RR'' \\ &- \Lambda^{-2}RR'\Lambda' + \tfrac{1}{2}\Lambda^{-1}R'^2 - \tfrac{1}{2}\Lambda - \tfrac{3}{2}\ell^{-2}\Lambda R^2, \end{aligned} \tag{2.7a}$$

$$H_r = P_RR' - \Lambda P_\Lambda' - \Gamma P_\Gamma', \tag{2.7b}$$

$$G = -P_\Gamma'. \tag{2.7c}$$

We have written the electric potential Φ in terms of the quantity

$$\tilde\Phi := \Phi - N^r\Gamma, \tag{2.8}$$

which now acts as the Lagrange multiplier associated with the Gauss constraint in Eq. (2.6). It would be possible to proceed retaining Φ as the Lagrange multiplier, and the supermomentum constraint would then be the same as without the electromagnetic field (see, for example, Ref. [49]). However, using $\tilde\Phi$ has the technical advantage that the supermomentum constraint (2.7b) generates spatial diffeomorphisms in both the gravitational and electromagnetic variables. This fact will prove useful in Sec. III.

The Hamiltonian equations of motion are obtained from local variations of Eq. (2.6). The constraint equations are

$$H = 0, \tag{2.9a}$$

$$H_r = 0, \tag{2.9b}$$

$$G = 0, \tag{2.9c}$$

and the dynamical equations of motion read

$$\dot\Lambda = N(R^{-2}\Lambda P_\Lambda - R^{-1}P_R) + (N^r\Lambda)', \tag{2.10a}$$

$$\dot R = -NR^{-1}P_\Lambda + N^rR', \tag{2.10b}$$

$$\dot\Gamma = N\Lambda R^{-2}P_\Gamma + (N^r\Gamma)' + \tilde\Phi', \tag{2.10c}$$

$$\begin{aligned} \dot P_\Lambda = {}&\tfrac{1}{2}N[-R^{-2}(P_\Lambda^2 + P_\Gamma^2) - (\Lambda^{-1}R')^2 + 1 + 3\ell^{-2}R^2] \\ &- \Lambda^{-2}N'RR' + N^rP_\Lambda', \end{aligned} \tag{2.10d}$$

$$\begin{aligned} \dot P_R = {}&N[\Lambda R^{-3}(P_\Lambda^2 + P_\Gamma^2) - R^{-2}P_\Lambda P_R - (\Lambda^{-1}R')' \\ &+ 3\ell^{-2}\Lambda R] - (\Lambda^{-1}N'R)' + (N^rP_R)', \end{aligned} \tag{2.10e}$$

$$\dot P_\Gamma = N^rP_\Gamma'. \tag{2.10f}$$

It is easy to verify that the Poisson brackets algebra of the constraints closes, and we thus have a first class constrained system [50].

We now wish to adopt boundary conditions that enforce every classical solution to be an exterior region of a RNAdS spacetime with a nondegenerate horizon (see Appendix A), such that the constant t hypersurfaces begin at the horizon bifurcation two-sphere at $r=0$ and reach the asymptotically anti–de Sitter infinity as $r\to\infty$.

Consider first the left end of the hypersurfaces. At $r \to 0$, we adopt the conditions

$$\Lambda(t,r) = \Lambda_0(t) + O(r^2), \tag{2.11a}$$

$$R(t,r) = R_0(t) + R_2(t) r^2 + O(r^4), \tag{2.11b}$$

$$P_\Lambda(t,r) = O(r^3), \tag{2.11c}$$

$$P_R(t,r) = O(r), \tag{2.11d}$$

$$N(t,r) = N_1(t) r + O(r^3), \tag{2.11e}$$

$$N^r(t,r) = N_1^r(t) r + O(r^3), \tag{2.11f}$$

$$\Gamma(t,r) = O(r), \tag{2.11g}$$

$$P_\Gamma(t,r) = Q_0(t) + Q_2(t) r^2 + O(r^4), \tag{2.11h}$$

$$\tilde{\Phi}(t,r) = \tilde{\Phi}_0(t) + O(r^2), \tag{2.11i}$$

where Λ_0 and R_0 are positive, and $N_1 \geqslant 0$. Here, $O(r^n)$ stands for a term whose magnitude at $r \to 0$ is bounded by r^n times a constant, and whose kth derivative at $r \to 0$ is similarly bounded by r^{n-k} times a constant for $1 \leqslant k \leqslant n$. It is straightforward to verify that these falloff conditions are consistent with the constraints $H = H_r = G = 0$, and that they are preserved by the time evolution equations. The metric falloff conditions (2.11a)–(2.11f), which are identical to those introduced in Ref. [32] in the context of the Schwarzschild black hole, guarantee that the classical solutions have a nondegenerate horizon, and that the constant t hypersurfaces begin at $r = 0$ at a horizon bifurcation two-sphere in a manner asymptotic to hypersurfaces of constant Killing time.[2] The coordinates become thus singular at $r \to 0$, but this singularity is quite precisely controlled. In particular, on a classical solution the future unit normal to a constant t hypersurface defines at $r \to 0$ a future timelike unit vector $n^a(t)$ at the bifurcation two-sphere, and the evolution of the constant t hypersurfaces boosts this vector according to

$$n^a(t_1) n_a(t_2) = -\cosh\left(\int_{t_1}^{t_2} \Lambda_0^{-1}(t) N_1(t) dt \right). \tag{2.12}$$

The falloff conditions (2.11g)–(2.11i) for the electromagnetic field variables are motivated by our thermodynamical goal, and they will be discussed further in Sec. V.

Consider then the right end of the hypersurfaces. At $r \to \infty$, we assume that the variables have asymptotic expansions in integer powers of $(1/r)$, with the leading order behavior

$$\Lambda(t,r) = \ell r^{-1} - \tfrac{1}{2}\ell^3 r^{-3} + \lambda(t) \ell^3 r^{-4} + O^\infty(r^{-5}), \tag{2.13a}$$

$$R(t,r) = r + \ell^2 \rho(t) r^{-2} + O^\infty(r^{-3}), \tag{2.13b}$$

$$P_\Lambda(t,r) = O^\infty(r^{-2}), \tag{2.13c}$$

$$P_R(t,r) = O^\infty(r^{-4}), \tag{2.13d}$$

$$N(t,r) = \Lambda^{-1} R' [\tilde{N}_+(t) + O^\infty(r^{-5})], \tag{2.13e}$$

$$N^r(t,r) = O^\infty(r^{-2}), \tag{2.13f}$$

$$\Gamma(t,r) = O^\infty(r^{-2}), \tag{2.13g}$$

$$P_\Gamma(t,r) = Q_+(t) + O^\infty(r^{-1}), \tag{2.13h}$$

$$\tilde{\Phi}(t,r) = \tilde{\Phi}_+(t) + O^\infty(r^{-1}), \tag{2.13i}$$

where $\tilde{N}_+(t) > 0$. $O^\infty(r^{-n})$ denotes a term that falls off at infinity as r^{-n}, and whose derivatives with respect to r fall off accordingly as r^{-n-k}, $k = 1,2, \ldots$. It is, again, straightforward to verify that these falloff conditions are consistent with the constraints and that they are preserved by the time evolution equations. Comparison with Ref. [47] shows that the metric is asymptotically anti–de Sitter, with the constant t hypersurfaces being asymptotic to hypersurfaces of constant Killing time, and $\tilde{N}_+(t)$ gives the rate at which the Killing time evolves with respect to t at the infinity. Note that the lapse-function N diverges at the infinity for any nonzero value of $\tilde{N}(t)$. For future use, we define the quantity

$$M_+(t) := \lambda(t) + 3\rho(t). \tag{2.14}$$

When the equations of motion hold, $M_+(t)$ is independent of t, and it is equal to the mass parameter of the RNAdS metric (A1).

Taken together, the falloff conditions (2.11) and (2.13) achieve our aim. Every classical solution is an exterior region of a RNAdS spacetime with a nondegenerate event horizon, such that the constant t hypersurfaces begin at the horizon bifurcation two-sphere and reach the asymptotically anti–de Sitter infinity. In particular, the classical solutions satisfy $R_2 > 0$.[3]

It would be possible to replace Eqs. (2.13a) and (2.13b) by

$$\Lambda(t,r) = \ell r^{-1} - \tfrac{1}{2}\ell^3 r^{-3} + O^\infty(r^{-4}), \tag{2.15a}$$

$$R(t,r) = r + O^\infty(r^{-2}), \tag{2.15b}$$

and then drop the assumption that the expansion proceeds in integer powers of $(1/r)$ beyond the order shown, provided one makes more precise assumptions about what is meant by the symbol O^∞. Alternatively, it would be possible to strengthen the falloff conditions to read

[2] The text in Ref. [32] contains at this point a minor inaccuracy. Equations (2.6a) and (2.6b) of Ref. [32] [our Eqs. (2.11a) and (2.11b)] are not sufficient to ensure that the hypersurfaces end at the horizon bifurcation two-sphere, but, for example, the set (2.6a)–(2.6c) of Ref. [32] [our Eqs. (2.11a)–(2.11c)] is.

[3] The falloff conditions (2.11) are compatible with either sign of R_2. The case $R_2 < 0$ would correspond to the bifurcation two-sphere of an inner horizon, which is excluded from the classical solutions only after the asymptotically anti–de Sitter falloff has been invoked at $r \to \infty$. If desired, the requirement $R_2 > 0$ could, of course, be already added to the conditions (2.11).

$$\Lambda(t,r)=\ell r^{-1}-\tfrac{1}{2}\ell^3 r^{-3}+\lambda(t)\ell^3 r^{-4}+O^\infty(r^{-4-\epsilon}), \tag{2.16a}$$

$$R(t,r)=r+O^\infty(r^{-2-\epsilon}), \tag{2.16b}$$

where $0<\epsilon\leqslant 1$, with similar changes in the rest of Eqs. (2.13). This would be analogous to the falloff conditions adopted for the asymptotically flat Schwarzschild case in Ref. [44], in that the value of the mass (2.14) could then be read solely from the expansion (2.16a) of Λ. One might also consider writing the theory in terms of a lapse function that has been rescaled by the factor $\Lambda^{-1}R'$: by Eq. (2.13e), the falloff of the new lapse at $r\to\infty$ would then be independent of the canonical variables. For concreteness, we shall adhere to the theory as written above.

We can now write an action principle compatible with our falloff conditions. Consider the total action

$$\begin{aligned} S[\Lambda,R,\Gamma,P_\Lambda,P_R,P_\Gamma;N,N^r,\tilde{\Phi}] &= S_\Sigma[\Lambda,R,\Gamma,P_\Lambda,P_R,P_\Gamma;N,N^r,\tilde{\Phi}] \\ &\quad + S_{\partial\Sigma}[\Lambda,R,Q_0,Q_+;N,\tilde{\Phi}_0,\tilde{\Phi}_+], \end{aligned} \tag{2.17}$$

where the boundary action is

$$S_{\partial\Sigma}[\Lambda,R,Q_0,Q_+;N,\tilde{\Phi}_0,\tilde{\Phi}_+]=\int dt(\tfrac{1}{2}R_0^2 N_1\Lambda_0^{-1}-\tilde{N}_+M_+ +\tilde{\Phi}_0 Q_0-\tilde{\Phi}_+Q_+). \tag{2.18}$$

The total action (2.17) is clearly well defined under our boundary conditions. Its variation contains a volume term proportional to the equations of motion, boundary terms from the initial and final hypersurfaces proportional to $\delta\Lambda$, δR, and $\delta\Gamma$, and boundary terms from $r=0$ and $r=\infty$ given by

$$\int dt[\tfrac{1}{2}R_0^2\delta(N_1\Lambda_0^{-1})-M_+\delta\tilde{N}_+ + Q_0\delta\tilde{\Phi}_0-Q_+\delta\tilde{\Phi}_+]. \tag{2.19}$$

The variation thus gives the desired classical equations of motion provided we fix, in addition to the initial and final values of Λ, R, and Γ, also the quantities $N_1\Lambda_0^{-1}$, $\tilde{N}_+$, $\tilde{\Phi}_0$, and $\tilde{\Phi}_+$. On a classical solution all these quantities have a clear geometrical interpretation. $N_1\Lambda_0^{-1}$ gives via Eq. (2.12) the evolution of the unit normal to the constant t hypersurface at the bifurcation two-sphere, and $\tilde{N}_+$ gives the evolution of the Killing time at the infinity. $\tilde{\Phi}_0$ and $\tilde{\Phi}_+$ describe the electromagnetic gauge in a way that will become more transparent in Sec. IV. Note from Eq. (2.11i) that when a classical solution is written in coordinates that are regular at the bifurcation two-sphere, the electromagnetic potential will be regular at the bifurcation two-sphere only if $\tilde{\Phi}_0=0$.

III. CANONICAL TRANSFORMATION

In this section we perform a canonical transformation, which generalizes that given in Ref. [44] for the spherically symmetric vacuum Einstein theory. Following Ref. [44], we first examine how the variables appearing in the action (2.17) carry the information about the geometry of the classical solution (A1). We then use this information as a guide for finding the canonical transformation.

A. Reconstruction

Under our boundary conditions, every classical solution is an exterior region of a RNAdS spacetime with a nondegenerate Killing horizon (see Appendix A). We now assume that we are given the canonical data $(\Lambda,R,\Gamma,P_\Lambda,P_R,P_\Gamma)$ on a spacelike hypersurface embedded in such a RNAdS spacetime. We wish to recover from the canonical data the mass and charge parameters of the spacetime, the information about the embedding of the hypersurface in the spacetime, and the information about the electromagnetic gauge.

Consider first the charge. The equations of motion imply that P_Γ is independent of both t and r. It is easily seen that in the curvature coordinates (A1), the value of P_Γ is just the charge Q. As P_Γ is unchanged by the gauge transformations generated by the constraints, it follows that in any gauge

$$Q=P_\Gamma. \tag{3.1}$$

Consider then the mass. The reconstruction of the function F appearing in the metric (A1a) proceeds exactly as in Ref. [44], with the result

$$F=\left(\frac{R'}{\Lambda}\right)^2-\left(\frac{P_\Lambda}{R}\right)^2. \tag{3.2}$$

From Eqs. (A1b) and (3.1), we find for the mass the expression

$$M=\frac{R}{2}\left(\frac{R^2}{\ell^2}+1+\frac{P_\Gamma^2}{R^2}-F\right), \tag{3.3}$$

where F is understood to be given by Eq. (3.2).

Consider then the embedding. By repeating the steps in Ref. [44], we obtain

$$-T'=R^{-1}F^{-1}\Lambda P_\Lambda, \tag{3.4}$$

which determines the embedding up to an overall additive constant in T. To determine the value of the additive constant, one needs to know the value of T at one point on the hypersurface.

Consider finally the electromagnetic gauge. By Eq. (A2), there exists a function $\xi(t,r)$ such that

$$A=R^{-1}Q\,dT+d\xi=(R^{-1}QT'+\xi')dr+(R^{-1}Q\dot{T}+\dot{\xi})dt. \tag{3.5}$$

From Eqs. (2.2), (3.1), and (3.4) we then obtain

$$\xi'=\Gamma+R^{-2}F^{-1}\Lambda P_\Lambda P_\Gamma, \tag{3.6}$$

which determines the value of ξ on the hypersurface up to an additive constant.

B. Transformation

We have seen that when the equations of motion hold, the quantities defined by Eqs. (3.1)–(3.4) have a transparent geometrical meaning. We now promote these equations into definitions of functions on the phase space, valid even when the equations of motion do not hold. Our aim is to complete the set of functions into a set that constitutes a canonical chart.

We shall from now on assume that the quantity R_2 in Eq. (2.11b) is positive. As noted in Sec. II, this is always the case for our classical solutions.

The functions M (3.3) and Q (3.1) Poisson commute with each other. The function $-T'$ (3.4) Poisson commutes with Q and is canonically conjugate to M. This suggests looking for a canonical transformation such that M and Q become two new coordinates, and $-T'$ becomes the momentum conjugate to M. As in the Schwarzschild case [44], the function $\mathsf{R}:=R$ Poisson commutes with M, Q, and $-T'$, and provides, therefore, a candidate for a new canonical coordinate. The crucial issue then is whether one can find momenta conjugate to Q and R such that the transformation is canonical.

A necessary condition for the prospective new momenta arises from the observation that the super-momentum constraint (2.7b) generates spatial diffeomorphisms in all the variables. Since M, Q, and R are spatial scalars, the expression for H_r in the new variables must be $P_M M' + P_Q Q' + P_{\mathsf{R}}\mathsf{R}'$. Equating this with Eq. (2.7b) and substituting for M and $P_M=-T'$ their expressions from Eqs. (3.3) and (3.4) gives only one equation for the two unknowns P_Q and P_{R}, but the structure of the equation as a linear combination of R' and P'_Γ suggests setting the coefficients of R' and P'_Γ individually to 0. These considerations suggest the transformation

$$M:=\tfrac{1}{2}R(R^2\ell^{-2}+1+P_\Gamma^2R^{-2}-F), \tag{3.7a}$$

$$P_M:=R^{-1}F^{-1}\Lambda P_\Lambda, \tag{3.7b}$$

$$\mathsf{R}:=R, \tag{3.7c}$$

$$\begin{aligned}P_{\mathsf{R}}:=&P_R-\tfrac{1}{2}R^{-1}\Lambda P_\Lambda-\tfrac{1}{2}R^{-1}F^{-1}\Lambda P_\Lambda\\&-R^{-1}\Lambda^{-2}F^{-1}[(\Lambda P_\Lambda)'(RR')-(\Lambda P_\Lambda)(RR')']\\&+\tfrac{1}{2}R^{-1}F^{-1}\Lambda P_\Lambda(P_\Gamma^2R^{-2}-3R^3\ell^{-2}),\end{aligned} \tag{3.7d}$$

$$Q:=P_\Gamma, \tag{3.7e}$$

$$P_Q:=-\Gamma-R^{-2}F^{-1}\Lambda P_\Lambda P_\Gamma, \tag{3.7f}$$

where F is given by Eq. (3.2). The analogy between the pairs (M,P_M) and (Q,P_Q) becomes manifest by observing from Eq. (3.6) that on a classical solution, P_Q carries the information about the electromagnetic gauge via $P_Q=-\xi'$.

It is now straightforward to demonstrate that the transformation (3.7) is indeed canonical. We begin with the identity

$$\begin{aligned}&P_\Lambda\delta\Lambda+P_R\delta R+P_\Gamma\delta\Gamma-P_M\delta M-P_{\mathsf{R}}\delta\mathsf{R}-P_Q\delta Q\\&=\left(\tfrac{1}{2}R\,\delta R\ln\left|\frac{RR'+\Lambda P_\Lambda}{RR'-\Lambda P_\Lambda}\right|\right)'\\&+\delta\left(\Gamma P_\Gamma+\Lambda P_\Lambda+\tfrac{1}{2}RR'\ln\left|\frac{RR'-\Lambda P_\Lambda}{RR'+\Lambda P_\Lambda}\right|\right),\end{aligned} \tag{3.8}$$

and integrate both sides with respect to r from $r=0$ to $r=\infty$. The first term on the right-hand side (RHS) gives substitution terms from $r=0$ to $r=\infty$ that vanish by virtue of our falloff conditions, and we obtain

$$\begin{aligned}\int_0^\infty dr(P_\Lambda\delta\Lambda+P_R\delta R+P_\Gamma\delta\Gamma)-\int_0^\infty(P_M\delta M+P_{\mathsf{R}}\delta\mathsf{R}\\+P_Q\delta_Q)=\delta\omega[\Lambda,R,\Gamma,P_\Lambda,P_\Gamma],\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\omega[\Lambda,R,\Gamma,P_\Lambda,P_\Gamma]=\int_0^\infty dr\Bigg(&\Gamma P_\Gamma+\Lambda P_\Lambda\\&+\tfrac{1}{2}RR'\ln\left|\frac{RR'-\Lambda P_\Lambda}{RR'+\Lambda R_\Lambda}\right|\Bigg).\end{aligned} \tag{3.10}$$

The functional $\omega[\Lambda,R,\Gamma,P_\Lambda,P_\Gamma]$ is well defined by virtue of the falloff conditions. Equations (3.9) and (3.10) show that the Liouville forms of the old and new variables differ only by an exact form, and the transformation is thus canonical.

The new variables have well-defined falloff properties at $r=0$ and $r\to\infty$. At $r=0$, Eqs. (2.11) imply

$$F(t,r)=4R_2^2\Lambda_0^{-2}r^2+O(r^4) \tag{3.11}$$

and

$$M(t,r)=M_0(t)+M_2(t)r^2+O(r^4), \tag{3.12a}$$

$$\mathsf{R}(t,r)=R_0(t)+R_2(t)r^2+O(r^4), \tag{3.12b}$$

$$Q(t,r)=Q_0(t)+Q_2(t)r^2+O(r^4), \tag{3.12c}$$

$$P_M(t,r)=O(r), \tag{3.12d}$$

$$P_{\mathsf{R}}(t,r)=O(r), \tag{3.12e}$$

$$P_Q(t,r)=O(r), \tag{3.12f}$$

where

$$M_0=\tfrac{1}{2}R_0(R_0^2\ell^{-2}+1+Q_0^2R_0^{-2}), \tag{3.13a}$$

$$\begin{aligned}M_2=&\tfrac{1}{2}R_2(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2}-4R_0R_2\Lambda_0^{-2})\\&+Q_0Q_2R_0^{-1}.\end{aligned} \tag{3.13b}$$

At $r\to\infty$, Eqs. (2.13) imply

$$M(t,r)=M_+(t)+O^\infty(r^{-1}), \tag{3.14a}$$

$$\mathsf{R}(t,r)=r+\ell^2\rho(t)r^{-2}+O^\infty(r^{-3}), \tag{3.14b}$$

$$Q(t,r)=Q_+(t)+O(r^{-1}), \tag{3.14c}$$

$$P_M(t,r)=O^\infty(r^{-6}), \tag{3.14d}$$

$$P_{\mathsf{R}}(t,r)=O^\infty(r^{-4}), \tag{3.14e}$$

$$P_Q(t,r)=O^\infty(r^{-2}), \tag{3.14f}$$

where $M_+(t)$ is given by Eq. (2.14).

The canonical transformation (3.7) becomes singular when $F=0$. Under our boundary conditions the classical solutions have $F>0$ for $r>0$. At the limit $r\to 0$, F approaches zero according to Eq. (3.11), but Eqs. (3.7) have still a well-defined limit obeying Eqs. (3.12). Our canonical transformation is, therefore, well defined and differentiable near the classical solutions, and similarly, the inverse transformation is well defined and differentiable near the classical solutions. From now on we shall assume that we are always in a neighborhood of the classical solutions such that $F>0$ holds for $r>0$.

C. Action

It is possible to write an action in the new variables by simply re-expressing the constraints (2.7) in terms of the new coordinates and momenta. A more transparent action can be found if we exercise the freedom to redefine the Lagrange multipliers.

The constraint terms in the bulk action (2.6) take the form

$$NH+N^rH_r+\tilde{\Phi}G=N^MM'+N^{\mathsf{R}}P_{\mathsf{R}}+N^QQ', \quad (3.15)$$

where

$$N^M=-NF^{-1}\Lambda^{-1}R'+N^rR^{-1}F^{-1}\Lambda P_\Lambda, \quad (3.16a)$$

$$N^{\mathsf{R}}=-NR^{-1}P_\Lambda+N^rR', \quad (3.16b)$$

$$N^Q=NR^{-1}F^{-1}\Lambda^{-1}R'P_\Gamma-N^r(\Gamma+R^{-2}F^{-1}\Lambda P_\Lambda P_\Gamma)-\tilde{\Phi}. \quad (3.16c)$$

When viewed as a linear transformation from $(N,N^r,\tilde{\Phi})$ to (N^M,N^{R},N^Q), Eqs. (3.16) are nonsingular for $r>0$. This suggests that we could take the constraint terms in the new bulk action to be those on the RHS of Eq. (3.15), with N^M, N^{R}, and N^Q as independent Lagrange multipliers. At $r\to\infty$, this would be satisfactory: Eqs. (3.16) imply the asymptotic behavior

$$N^M(t,r)=-\tilde{N}_+(t)+O^\infty(r^{-5}), \quad (3.17a)$$

$$N^{\mathsf{R}}(t,r)=O^\infty(r^{-2}), \quad (3.17b)$$

$$N^Q(t,r)=-\tilde{\Phi}_+(t)+O^\infty(r^{-1}), \quad (3.17c)$$

and one could then fix $\tilde{N}_+(t)$ and $\tilde{\Phi}_+(t)$ as in Sec. II after adding the boundary action

$$-\int dt(\tilde{N}_+M_++\tilde{\Phi}_+Q_+). \quad (3.18)$$

However, at $r=0$ we have

$$N^M(t,r)=-\tfrac{1}{2}N_1\Lambda_0R_2^{-1}+O(r^2), \quad (3.19a)$$

$$N^{\mathsf{R}}(t,r)=O(r^2), \quad (3.19b)$$

$$N^Q(t,r)=-\tilde{\Phi}_0(t)+\tfrac{1}{2}N_1\Lambda_0Q_0R_2^{-1}R_0^{-1}+O(r^2), \quad (3.19c)$$

which says that fixing N^M and N^Q at $r=0$ to values that are independent of the canonical variables is not equivalent to fixing $N_1\Lambda_0^{-1}$ and $\tilde{\Phi}_0$ to values that are independent of the canonical variables. We, therefore, need to redefine N^M and N^Q near $r=0$, without affecting their behavior at $r\to\infty$.

To proceed, we make two assumptions. First, we assume $M_0>M_{\rm crit}(Q_0)$, where the function $M_{\rm crit}$ is defined in Appendix A. Second, we regard Eq. (3.13a) as defining R_0 in terms of M_0 and Q_0 as $R_0=R_{\rm hor}(M_0,Q_0)$, where the function $R_{\rm hor}$ is defined in Appendix A. As discussed in Appendix A, these assumptions are always true for our classical solutions, and they, therefore, merely tighten the neighborhood of the classical solutions in whch the field variables may take values. For future use, we note that these assumptions imply $3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2}>0$, and the variation of R_0 takes the form

$$\delta R_0=2(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2})^{-1}(\delta M_0-R_0^{-1}Q_0\delta Q_0). \quad (3.20)$$

Define now the quantities $\tilde{N}$ and $\tilde{N}^Q$ by

$$N^M=-\tilde{N}^M[(1-g)+2gR_0(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2})^{-1}], \quad (3.21a)$$

$$N^Q=2\tilde{N}^MgQ_0(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2})^{-1}-\tilde{N}^Q, \quad (3.21b)$$

where $g(r)$ is a smooth decreasing function that vanishes at $r\to\infty$ as $O^\infty(r^{-5})$, and approaches the value 1 at $r\to 0$ as $g(r)=1+O(r^2)$. Equations (3.21) then define a nonsingular linear transformation from (N^M,N^{R}) to $(\tilde{N}^M,\tilde{N}^Q)$. The asymptotic behavior at $r\to\infty$ is

$$\tilde{N}^M(t,r)=\tilde{N}_+(t)+O^\infty(r^{-5}), \quad (3.22a)$$

$$\tilde{N}^Q(t,r)=\tilde{\Phi}_+(t)+O^\infty(r^{-1}), \quad (3.22b)$$

and the asymptotic behavior at $r=0$ is

$$\tilde{N}^M(t,r)=\tilde{N}_0^M(t)+O(r^2), \quad (3.23a)$$

$$\tilde{N}^Q(t,r)=\tilde{\Phi}_0(t)+O(r^2), \quad (3.23b)$$

where

$$\tilde{N}_0^M=\tfrac{1}{4}N_1\Lambda_0R_0^{-1}R_2^{-1}(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2}). \quad (3.24)$$

When the constraints $M'=0$ and $Q'=0$ hold, Eqs. (3.12a), (3.12c), and (3.13b) show that

$$\tilde{N}_0^M \overset{\substack{Q'=0\\M'=0}}{=} N_1\Lambda_0^{-1}. \quad (3.25)$$

Thus, when the constraints hold, fixing $\tilde{N}^M$ and $\tilde{N}^Q$ at $r=0$ is equivalent to fixing $N_1\Lambda_0^{-1}$ and $\tilde{\Phi}_0$. We, therefore, adopt $\tilde{N}^M$, N^{R}, and $\tilde{N}^Q$ as a set of new independent Lagrange multipliers.

The bulk action takes the form

$$S_\Sigma[M,\mathsf{R},Q,P_M,P_\mathsf{R},P_Q;\tilde{N}^M,N^\mathsf{R},\tilde{N}^Q]$$
$$=\int dt\int_0^\infty dr\{(P_M\dot{M}+P_\mathsf{R}\dot{\mathsf{R}}+P_Q\dot{Q}+\tilde{N}^Q Q'-N^\mathsf{R}P_\mathsf{R}+\tilde{N}^M[(1-g)M'+2g(3R_0^2\ell^{-2}+1-Q_0^2R_0^{-2})^{-1}(R_0M'-Q_0Q')]\}. \quad (3.26)$$

The total action is taken to be

$$S[M,\mathsf{R},Q,P_M,P_\mathsf{R},P_Q;\tilde{N}^M,N^\mathsf{R},\tilde{N}^Q]=S_\Sigma[M,\mathsf{R},Q,P_M,P_\mathsf{R},P_Q;\tilde{N}^M,N^\mathsf{R},\tilde{N}^Q]+S_{\partial\Sigma}[M_0,M_+,Q_0,Q_+;\tilde{N}_+,\tilde{\Phi}_0,\tilde{\Phi}_+], \quad (3.27)$$

where

$$S_{\partial\Sigma}[M_0,M_+,Q_0,Q_+;\tilde{N}_+,\tilde{\Phi}_0,\tilde{\Phi}_+]=\int dt(\tfrac{1}{2}R_0^2\tilde{N}_0^M-\tilde{N}_+M_++\tilde{\Phi}_0Q_0-\tilde{\Phi}_+Q_+). \quad (3.28)$$

The quantities to be varied independently are M, R, Q, P_M, P_R, P_Q, $\tilde{N}^M$, N^R, and $\tilde{N}^Q$, and the boundary conditions for the new Lagrange multipliers are given by Eqs. (3.17b), (3.19b), (3.22), and (3.23). The volume term in the variation of Eq. (3.27) is proportional to the equations of motion

$$\dot{M}=0, \quad (3.29a)$$
$$\dot{\mathsf{R}}=N^\mathsf{R}, \quad (3.29b)$$
$$\dot{Q}=0, \quad (3.29c)$$
$$\dot{P}_M=(N^M)', \quad (3.29d)$$
$$\dot{P}_\mathsf{R}=0, \quad (3.29e)$$
$$\dot{P}_Q=(N^Q)', \quad (3.29f)$$
$$M'=0, \quad (3.29g)$$
$$P_\mathsf{R}=0, \quad (3.29h)$$
$$Q'=0, \quad (3.29i)$$

where N^M and N^Q are now understood to be defined by Eqs. (3.21). The boundary terms in the variation consist of terms proportional to δM, $\delta\mathsf{R}$, and δQ on the initial and final hypersurfaces, and terms from $r=0$ and $r=\infty$ given by

$$\int dt(\tfrac{1}{2}R_0^2\delta\tilde{N}_0^M-M_+\delta\tilde{N}_++Q_0\delta\tilde{\Phi}_0-Q_+\delta\tilde{\Phi}_+). \quad (3.30)$$

To arrive at Eq. (3.30), Eq. (3.20) has been used. The action (3.27) thus yields the equations of motion (3.29) provided that we fix, in addition to the initial and final values of the new canonical coordinates, also the quantities $\tilde{N}_0^M$, $\tilde{N}_+$, $\tilde{\Phi}_0$, and $\tilde{\Phi}_+$. Because of Eq. (3.25), these fixed quantities at the right and left ends have precisely the same interpretation in terms of the geometry of the classical solutions as the fixed quantities in the action (2.17).

IV. HAMILTONIAN REDUCTION

In this section we shall reduce the action (3.27) to the true dynamical degrees of freedom by solving the constraints.

The constraints (3.29g) and (3.29i) imply that M and Q are independent of r. We can, therefore, write

$$M(t,r)=\mathbf{m}(t), \quad (4.1a)$$
$$Q(t,r)=\mathbf{q}(t). \quad (4.1b)$$

Substituting these and the constraint (3.29h) back into Eq. (3.27) yields the true Hamiltonian action

$$S[\mathbf{m},\mathbf{q},\mathbf{p_m},\mathbf{p_q};\tilde{N}_0^M,\tilde{N}_+,\tilde{\Phi}_+,\tilde{\Phi}_0]=\int dt(\mathbf{p_m}\dot{\mathbf{m}}+\mathbf{p_q}\dot{\mathbf{q}}-\mathbf{h}), \quad (4.2)$$

where

$$\mathbf{p_m}=\int_0^\infty dr P_M, \quad (4.3a)$$
$$\mathbf{p_q}=\int_0^\infty dr P_Q. \quad (4.3b)$$

The reduced Hamiltonian $\mathbf{h}$ in Eq. (4.2) is

$$\mathbf{h}=-\tfrac{1}{2}R_\mathrm{h}^2\tilde{N}_0^M+\tilde{N}_+\mathbf{m}+(\tilde{\Phi}_+-\tilde{\Phi}_0)\mathbf{q}, \quad (4.4)$$

where $R_\mathrm{h}:=R_\mathrm{hor}(\mathbf{m},\mathbf{q})$. The assumptions made in the previous section imply

$$\mathbf{m}>M_\mathrm{crit}(\mathbf{q}), \quad (4.5)$$

and $\mathbf{h}$ is, therefore, well defined. Note that $\mathbf{h}$ is, in general, explicitly time dependent through the prescribed functions $\tilde{N}_0^M(t)$, $\tilde{N}_+(t)$, $\tilde{\Phi}_+(t)$, and $\tilde{\Phi}_0(t)$.

The variational principle associated with the reduced action (4.2) fixes the initial and final values of the coordinates $\mathbf{m}$ and $\mathbf{q}$. The equations of motion are

$$\dot{\mathbf{m}}=0, \quad (4.6a)$$
$$\dot{\mathbf{q}}=0, \quad (4.6b)$$
$$\dot{\mathbf{p}}_\mathbf{m}=2R_\mathrm{h}(3R_\mathrm{h}^2\ell^{-2}+1-\mathbf{q}^2R_\mathrm{h}^{-2})^{-1}\tilde{N}_0^M-\tilde{N}_+, \quad (4.6c)$$
$$\dot{\mathbf{p}}_\mathbf{q}=-2\mathbf{q}(3R_\mathrm{h}^2\ell^{-2}+1-\mathbf{q}^2R_\mathrm{h}^{-2})^{-1}\tilde{N}_0^M+\tilde{\Phi}_0-\tilde{\Phi}_+. \quad (4.6d)$$

Equations (4.6a) and (4.6b) are readily understood in terms of the statement that on a classical solution $\mathbf{m}$ and $\mathbf{q}$ are, respectively, equal to the mass and charge parameters of the RNAdS solution. To understand Eq. (4.6c), recall from Sec. III that on a classical solution $P_M=-T'$, where T is the Killing time. From Eq. (4.3a) we see that $\mathbf{p_m}=T_0-T_+$, where T_0 and T_+ are, respectively, the values of T at the left and right ends of the constant t hypersurface. As the constant t hypersurface evolves in the RNAdS spacetime, the first and second term on the RHS of Eq. (4.6c) are, respectively, equal to $\dot{T}_0$ and $-\dot{T}_+$. The interpretation of Eq. (4.6d) is analogous. On a classical solution we have $\mathbf{p_q}=\xi_0-\xi_+$, where ξ is the function that specifies the electromagnetic gauge via Eq. (3.5). The first two terms on the RHS of Eq. (4.6d) give $\dot{\xi}_0$, and the last term gives $-\dot{\xi}_+$.

V. QUANTUM THEORY AND THE GRAND PARTITION FUNCTION

We shall now quantize the reduced Hamiltonian theory of Sec. IV. Our aim is to construct the time evolution operator in the Hamiltonian quantum theory, and then to obtain a grand partition function via an analytic continuation of this operator.

A. Quantization

As is well known, the quantization of a given classical Hamiltonian theory requires input [51–53], and the questions of physically appropriate input for a quantum black hole remain largely open. For the purposes of the present paper we shall be content to define the quantum theory in essence by fiat, following Refs. [32,44]. Our main physical conclusions will emerge from the semiclassical regime of the theory, and at this level one may reasonably hope the details of the quantization not to be crucial.

We regard $\mathbf{m}$ and $\mathbf{q}$ as configuration variables, satisfying the inequality (4.5). The wave functions are of the form $\psi(\mathbf{m},\mathbf{q})$, and the inner product is taken to be

$$(\psi,\chi)=\int_A \mu d\mathbf{m}d\mathbf{q}\overline{\psi}\chi, \tag{5.1}$$

where $A\subset\mathbb{R}^2$ is the domain (4.5) and $\mu(\mathbf{m},\mathbf{q})$ is a smooth positive weight factor. The Hilbert space is thus $\mathcal{H}:=L^2(A;\mu d\mathbf{m}d\mathbf{q})$. We assume that μ is a slowly varying function, in a sense to be made more precise later, but otherwise it will remain arbitrary.

The Hamiltonian operator $\hat{\mathbf{h}}(t)$ is taken to act as pointwise multiplication by the function $\mathbf{h}(\mathbf{m},\mathbf{q};t)$ (4.4): $\psi(\mathbf{m},\mathbf{q})\mapsto\mathbf{h}(\mathbf{m},\mathbf{q};t)\psi(\mathbf{m},\mathbf{q})$. $\hat{\mathbf{h}}(t)$ is an unbounded essentially self-adjoint operator [54], and the corresponding unitary time evolution operator in $\mathcal{H}$ is

$$\hat{K}(t_2;t_1)=\exp\left[-i\int_{t_1}^{t_2}dt'\hat{\mathbf{h}}(t')\right]. \tag{5.2}$$

$\hat{K}(t_2;t_1)$ acts in $\mathcal{H}$ by pointwise multiplication by the function

$$K(\mathbf{m},\mathbf{q};\mathcal{T},\Xi_+,\Xi_0,\Theta)=\exp[-i\mathbf{m}\mathcal{T}-i\mathbf{q}(\Xi_+-\Xi_0)+\tfrac{1}{2}iR_h^2\Theta], \tag{5.3}$$

where

$$\mathcal{T}:=\int_{t_1}^{t_2}dt\tilde{N}_+(t), \tag{5.4a}$$

$$\Xi_+:=\int_{t_1}^{t_2}dt\tilde{\Phi}_+(t), \tag{5.4b}$$

$$\Xi_0:=\int_{t_1}^{t_2}dt\tilde{\Phi}_0(t), \tag{5.4c}$$

$$\Theta:=\int_{t_1}^{t_2}dt\tilde{N}_0^M(t). \tag{5.4d}$$

$\hat{K}(t_2;t_1)$, therefore, depends on t_1 and t_2 only through the quantities on the left-hand side of Eqs. (5.4), and we may write $\hat{K}(t_2;t_1)$ as $\hat{K}(\mathcal{T},\Xi_+,\Xi_0,\Theta)$. The composition law, $\hat{K}(t_3;t_2)\hat{K}(t_2;t_1)=\hat{K}(t_3;t_1)$, amounts to independent addition in each of the four parameters in $\hat{K}(\mathcal{T},\Xi_+,\Xi_0,\Theta)$, and we may regard these four parameters as independent evolution parameters specified by the boundary conditions. $\mathcal{T}$ is the Killing time elapsed at the infinity, and Θ is the boost parameter elapsed at the bifurcation two-sphere. Ξ_+ and Ξ_0 can be computed from the line integral of the electromagnetic potential (2.2) along the timelike curve of constant r and constant angular variables as this curve approaches, respectively, the infinity and the bifurcation two-sphere.

B. Grand partition function

We shall now construct a grand partition function by continuing the time evolution operator to imaginary time and taking the trace. We begin by discussing the boundary conditions for the relevant thermodynamical ensemble.

The envisaged semiclassical thermodynamical situation consists of a charged spherically symmetric black hole in an asymptotically anti–de Sitter space, in thermal equilibrium with a bath of Hawking radiation. If the back reaction from the radiation is neglected, the geometry is described by the RNAdS metric (A1). Assuming that the local temperature is given in the usual manner in terms of the surface gravity and the redshift factor [18,35], we see that the local temperature is $F^{-1/2}\beta^{-1}$, where F is given by Eq. (A1b) and

$$\beta:=4\pi R_0(3\ell^{-2}R_0^2+1-Q^2R_0^{-2})^{-1}. \tag{5.5}$$

At the infinity the local temperature vanishes as $\beta^{-1}\ell R^{-1}[1+O^\infty(\ell^2R^{-2})]$, and β^{-1} can thus be extracted from the asymptotic behavior as the coefficient of the leading order term ℓR^{-1}. We shall follow Refs. [18,35] and regard β^{-1} as a renormalized temperature at infinity.

The electromagnetic variable with thermodynamic interest for us is the electric potential difference between the horizon and the infinity, in the curvature coordinates (A1) and in an electromagnetic gauge that makes A invariant under the Killing time translations. We denote this quantity by ϕ. From Eq. (A2) it is seen that on a classical solution $\phi=QR_0^{-1}$.

We shall consider a thermodynamical ensemble in which the fixed quantities are β and ϕ. This data can be interpreted as that for a grand canonical ensemble, with ϕ being analogous to the chemical potential [7,8,26]. Our aim is to obtain a grand partition function $\mathcal{Z}(\beta,\phi)$ by continuing the time evolution operator of the Lorentzian Hamiltonian theory to imaginary time and taking the trace.

The continuation of $\mathcal{T}$ is straightforward: comparing the definition of β to the falloff of N in Eq. (2.13) and to the definition (5.4a), we are led to set $\mathcal{T}=-i\beta$. For the continuation of Θ we choose $\Theta=-2\pi i$, motivated by the regularity of the classical Euclidean solutions as in Ref. [32]. We mentioned at the end of Sec. II that the regularity of the electromagnetic potential at the bifurcation two-sphere of the Lorentzian solutions requires $\tilde{\Phi}_0=0$; similarly, requiring regularity of the electromagnetic potential at the horizon of the classical Euclidean solutions now leads us to set $\Xi_0=0$. Finally, recall that Ξ_+ gives the constant r line integral of the electromagnetic potential (2.2) at the infinity. Comparing this to the definition of ϕ, we set $\Xi_+=-\mathcal{T}\phi=i\beta\phi$. We are thus led to propose for the grand partition function the expression

$$\mathcal{Z}(\beta,\phi)=\mathrm{Tr}[\hat{K}(-i\beta,i\beta\phi,0,-2\pi i)]. \tag{5.6}$$

As it stands, the trace in Eq. (5.6) is divergent, but one can argue as in Refs. [32,34] that a suitable regularization and renormalization yields the result

$$\mathcal{Z}_{\mathrm{ren}}(\beta,\phi)=\mathcal{N}\int_A \mu\, d\mathbf{m}\, d\mathbf{q} \exp[-\beta(\mathbf{m}-\mathbf{q}\phi)+\pi R_{\mathrm{h}}^2], \tag{5.7}$$

where we have substituted for K the explicit expression (5.3). The normalization factor $\mathcal{N}$ may depend on ℓ, but we shall assume that it does not depend on β or on ϕ.

Provided the weight factor μ is slowly varying compared with the exponential in Eq. (5.7), it is easy to verify, using the definition of R_{h} given after Eq. (4.4), that the integral in Eq. (5.7) is convergent. Equation (5.7) thus yields a well-defined grand partition function. Comparing with ordinary pressure-volume-temperature systems [21], ϕ is now indeed seen to be analogous to the chemical potential, and the quantities $\mathbf{m}$ and $\mathbf{q}$ are, respectively, analogous to the energy and the particle number. We shall examine the thermodynamical properties of this grand partition function in the next section.

VI. THERMODYNAMICS IN THE GRAND CANONICAL ENSEMBLE

It is useful to change the integration variables in Eq. (5.7) from the pair $(\mathbf{m},\mathbf{q})$ to the pair $(R_{\mathrm{h}},\mathbf{q})$. From Eq. (A6) we obtain

$$\mathbf{m}=\tfrac{1}{2}R_{\mathrm{h}}(R_{\mathrm{h}}^2\ell^{-2}+1+\mathbf{q}^2R_{\mathrm{h}}^{-2}), \tag{6.1}$$

and the grand partition function takes the form

$$\mathcal{Z}_{\mathrm{ren}}(\beta,\phi)=\mathcal{N}\int_{A'} \tilde{\mu}\, dR_{\mathrm{h}}\, d\mathbf{q} \exp(-I_*), \tag{6.2}$$

where

$$I_*(R_{\mathrm{h}},\mathbf{q}):=\tfrac{1}{2}\beta R_{\mathrm{h}}(R_{\mathrm{h}}^2\ell^{-2}+1+\mathbf{q}^2R_{\mathrm{h}}^{-2})-\beta\phi\mathbf{q}-\pi R_{\mathrm{h}}^2. \tag{6.3}$$

One may view I_* as an effective action or as a reduced action [24–26]. The integration domain A' is given by the inequalities

$$0\leqslant R_{\mathrm{h}}, \tag{6.4a}$$

$$\mathbf{q}^2\leqslant R_{\mathrm{h}}^2(1+3R_{\mathrm{h}}^2\ell^{-2}), \tag{6.4b}$$

and the weight factor $\tilde{\mu}$ is obtained from μ by including the Jacobian $|\partial(\mathbf{m},\mathbf{q})/\partial(R_{\mathrm{h}},\mathbf{q})|$. Note that because of Eq. (6.4b), I_* remains finite as $R_{\mathrm{h}}\to 0$.

As $\tilde{\mu}$ is assumed to be slowly varying, we can estimate $\mathcal{Z}_{\mathrm{ren}}(\beta,\phi)$ by the saddle point approximation to Eq. (6.2). For this, we need to find the critical points of I_* in the interior of A'.

When $\phi^2<1-\frac{4}{3}\pi^2\ell^2\beta^{-2}$, I_* has no critical point. When $1-\frac{4}{3}\pi^2\ell^2\beta^{-2}<\phi^2<1$, the two critical points of I_* are at

$$R_{\mathrm{h}}=R_{\mathrm{h}}^{\pm}:=\frac{2\pi\ell^2}{3\beta}\left(1\pm\sqrt{1+\frac{3\beta^2(\phi^2-1)}{4\pi^2\ell^2}}\right), \tag{6.5a}$$

$$\mathbf{q}=\mathbf{q}^{\pm}:=\phi R_{\mathrm{h}}^{\pm}. \tag{6.5b}$$

The lower signs do not give a local extremum, but the upper signs give a local minimum. In the limiting case $1-\frac{4}{3}\pi^2\ell^2\beta^{-2}=\phi^2$, the only critical point is $(R_{\mathrm{h}}^+,\mathbf{q}^+)$, but it is not a local extremum. Finally, when $\phi^2\geqslant 1$, the only critical point is $(R_{\mathrm{h}}^+,\mathbf{q}^+)$, and it is a local minimum. Whenever the critical points exist, the value of I_* at these points can be written as

$$I_*(R_{\mathrm{h}}^{\pm},\mathbf{q}^{\pm})=\frac{\pi(R_{\mathrm{h}}^{\pm})^2[1-\phi^2-(R_{\mathrm{h}}^{\pm})^2\ell^{-2}]}{1-\phi^2+3(R_{\mathrm{h}}^{\pm})^2\ell^{-2}}. \tag{6.6}$$

As I_* grows without bound in the noncompact directions in A', the global minimum can be found by examining I_* at the critical points and on the boundary of A'. When $\phi^2>1-\pi^2\ell^2\beta^{-2}$, the global minimum is at the critical point $(R_{\mathrm{h}}^+,\mathbf{q}^+)$, and $I_*(R_{\mathrm{h}}^+,\mathbf{q}^+)$ is negative. When $\phi^2<1-\pi^2\ell^2\beta^{-2}$, the global minimum is at $R_{\mathrm{h}}=0=\mathbf{q}$, where I_* vanishes. In the limiting case $\phi^2=1-\pi^2\ell^2\beta^{-2}$, I_* vanishes at $(R_{\mathrm{h}}^+,\mathbf{q}^+)$ and at $R_{\mathrm{h}}=0=\mathbf{q}$, but is positive everywhere else.

We thus see that for $\phi^2>1-\pi^2\ell^2\beta^{-2}$, $\mathcal{Z}_{\mathrm{ren}}$ can be approximated as

$$\mathcal{Z}_{\mathrm{ren}}(\beta,\phi)\approx P\exp[-I_*(R_{\mathrm{h}}^+,\mathbf{q}^+)], \tag{6.7}$$

where P is a slowly varying prefactor. The approximation becomes presumably progressively better with increasing $|I_*(R_{\mathrm{h}}^+,\mathbf{q}^+)|$. For $\phi^2<1-\pi^2\ell^2\beta^{-2}$, the dominant contribution to $\mathcal{Z}_{\mathrm{ren}}$ comes from the vicinity of $R_{\mathrm{h}}=0=\mathbf{q}$, and the behavior of $\mathcal{Z}_{\mathrm{ren}}$ depends more sensitively on the weight factor $\tilde{\mu}$.

These results for $\mathcal{Z}_{\mathrm{ren}}$ are consistent with what one would expect just from the existence of (Lorentzian) black hole solutions under fixing ϕ and the renormalized inverse Hawking temperature β (5.5). It can be verified that such solutions exist precisely at the critical points of I_*: the values of $\mathbf{m}$ and $\mathbf{q}$ at these critical points are just the mass and charge parameters of the black hole. Further, the value of I_* at a critical point is simply the Euclidean action of the corresponding Euclideanized black hole solution. When a unique classical solution exists, it dominates the grand partition function; when two distinct classical solutions exist, the grand partition function is dominated either by the larger mass classical solution or by none of the classical solutions. The situation is thus remarkably similar to that found in the absence of a cosmological constant when the boundary conditions are set on a finite size box [26].

Let us now consider the thermodynamical predictions from $\mathcal{Z}_{\rm ren}$. Recall that the thermal expectation values of the energy and charge in the grand canonical ensemble are given by

$$\langle E\rangle = \left(-\frac{\partial}{\partial\beta}+\beta^{-1}\phi\frac{\partial}{\partial\phi}\right)(\ln\mathcal{Z}_{\rm ren}), \tag{6.8a}$$

$$\langle Q\rangle = \beta^{-1}\frac{\partial(\ln\mathcal{Z}_{\rm ren})}{\partial\phi}. \tag{6.8b}$$

When $\mathcal{Z}_{\rm ren}$ is dominated by the critical point $(R_{\rm h}^{+},\mathbf{q}^{+})$, we find

$$\langle E\rangle \approx \mathbf{m}^{+}, \tag{6.9a}$$

$$\langle Q\rangle \approx \mathbf{q}^{+}, \tag{6.9b}$$

where $\mathbf{m}^{+}$ is obtained from $(R_{\rm h}^{+},\mathbf{q}^{+})$ through Eq. (6.1). That is, the thermal expectation values of the energy and the charge are simply the mass and charge parameters of the dominant classical solution. In particular, there is no additional contribution to the mass from the gravitational binding energy associated with the thermal energy, or from the electrostatic binding energy associated with the charge. Such additional, finite-size contributions were found to be present in the finite-size ensembles of Refs. [14,24,26,34].

It is easily seen that $(\partial\mathbf{m}^{+}/\partial\beta)<0$. This means that when the approximation (6.9a) is good, the (constant ϕ) heat capacity, $C_{\phi}=-\beta^{2}(\partial\langle E\rangle/\partial\beta)$, is positive. In the regime (6.9a), the system is thus stable under thermal fluctuations in the energy. Note that as $(\partial\mathbf{m}^{-}/\partial\beta)>0$, a grand partition function dominated by the lower mass classical solution would be thermodynamically unstable [25]. This is analogous to what happens in the absence of a cosmological constant under the boxed boundary conditions considered in Refs. [14,24,26].

It is also easily seen that $(\partial\mathbf{q}^{+}/\partial\phi)>0$. This shows that when the approximation (6.9b) is good, we have $(\partial\langle Q\rangle/\partial\phi)>0$, and the system is stable under thermal fluctuations in the charge. More generally, one can show directly from the expressions (6.2), (6.3), and (6.8b) that $(\partial\langle Q\rangle/\partial\phi)>0$ holds always, even when the approximation (6.9b) is not good.

The entropy in the grand canonical ensemble is given by

$$S=\left(1-\beta\frac{\partial}{\partial\beta}\right)(\ln\mathcal{Z}_{\rm ren}). \tag{6.10}$$

When the approximation (6.7) is good, we have $S\approx\pi(R_{\rm h}^{+})^{2}$, which means that the entropy is one quarter of the horizon area. This is the anticipated Bekenstein-Hawking result.

Finally, when $\mathcal{Z}_{\rm ren}$ is not dominated by a critical point, the thermodynamical predictions become much more sensitive to the choice of the weight factor $\tilde{\mu}$. As in Refs. [24–26,34], one can view the transition in the qualitative behavior of $\mathcal{Z}_{\rm ren}$ as evidence for a phase transition between a black hole sector and a topologically different sector of the theory; in the case at hand, the second sector might be referred to as ''hot anti–de Sitter space.'' On classical grounds one might have expected this transition to occur near $\phi^{2}\approx 1-\frac{4}{3}\pi^{2}\ell^{2}\beta^{-2}$, where the classical solutions disappear. However, we saw that the transition in fact occurs near $\phi^{2}\approx 1-\pi^{2}\ell^{2}\beta^{-2}$, where the two classical solutions still exist. This is highly similar to what happens in four dimensions under boxed boundary conditions without a cosmological constant [24,26], but subtly different from what happens in two dimensions with Witten's dilatonic black hole [34].

VII. THE CANONICAL ENSEMBLE

We have seen that the Hamiltonian formulation of Secs. II–IV led into a thermodynamical grand canonical ensemble where the fixed quantities are the renormalized inverse temperature β at infinity and the electric potential difference ϕ between the horizon and the infinity. From the thermodynamical viewpoint, another natural ensemble for the charged black hole in asymptotically anti–de Sitter space is the canonical ensemble, where one allows fluctuations in ϕ but fixes instead the charge $\mathbf{q}$. In this section we shall outline the recovery of the canonical ensemble from a Lorentzian Hamiltonian analysis, and briefly discuss the thermodynamical properties of the black hole in this ensemble.

As a starting point, we modify the boundary conditions of the Hamiltonian theory of Sec. II by leaving $\tilde{\Phi}_{0}(t)$ and $\tilde{\Phi}_{+}(t)$ unspecified but fixing $Q_{0}(t)$ and $Q_{+}(t)$ to be prescribed functions of t. The action is obtained from Eqs. (2.17) and (2.18) by omitting the terms $\int dt(\tilde{\Phi}_{0}Q_{0}-\tilde{\Phi}_{+}Q_{+})$ from Eq. (2.18). Clearly, classical solutions exist only when $Q_{0}(t)$ and $Q_{+}(t)$ are chosen independent of t and are equal. We shall from now on assume that the boundary data is chosen in this manner.

One way to proceed is simply to push through the canonical transformation of Sec. III, noting that the new boundary conditions merely result into minor modifications. It is only when one subsequently performs a Hamiltonian reduction along the lines of Sec. IV that the new boundary conditions give rise to important differences. First, the boundary data for $Q_{0}(t)$ and $Q_{+}(t)$ imply that the quantity $\mathbf{q}(t)$ defined by Eq. (4.1b) is a t-independent constant whose value is completely determined by the boundary conditions. Therefore, the Liouville term $\int dt\mathbf{p}_{\mathbf{q}}\dot{\mathbf{q}}$ drops entirely out of the action (4.2). Second, because of the terms that were omitted from the boundary action (2.18), the term $(\tilde{\Phi}_{+}-\tilde{\Phi}_{0})\mathbf{q}$ drops out of the reduced Hamiltonian (4.4). This means that in the reduced Hamiltonian theory $\mathbf{q}$ has become an external parameter specified by the boundary conditions: it is not varied in the action, and does not have a conjugate momentum. The new reduced action reads

$$S_{C}[\mathbf{m},\mathbf{p}_{\mathbf{m}};\tilde{N}_{0}^{M},\tilde{N}_{+};\mathbf{q}]=\int dt(\mathbf{p}_{\mathbf{m}}\dot{\mathbf{m}}-\mathbf{h}_{C}), \tag{7.1}$$

where

$$\mathbf{h}_{C}=-\tfrac{1}{2}R_{\rm h}^{2}\tilde{N}_{0}^{M}+\tilde{N}_{+}\mathbf{m}. \tag{7.2}$$

Here, $R_h := R_{hor}(\mathbf{m},\mathbf{q})$ as before, and the assumptions made in the canonical transformation again imply that Eq. (4.5) holds.

An alternative way to proceed under the new boundary data is to partially reduce the action already in the variables of Sec. II by solving the constraint (2.9c). One uses the constraint (2.9c) and the equation of motion (2.10f) to set $P_\Gamma(t,r)$ equal to the constant specified in the boundary data, and substitutes this back in the action. The Liouville term $\int_0^\infty dr P_\Gamma \dot{\Gamma}$ then becomes a total time derivative and can be dropped. One thus obtains an action that no longer involves Γ or $\tilde{\Phi}$, involves P_Γ only as a prescribed constant, and correctly yields the equations of motion for the remaining variables. One can now perform a canonical transformation from the variables $(\Lambda,R,P_\Lambda,P_R)$ to the new variables $(M,\mathsf{R},P_M,P_{\mathsf{R}})$, defined as in Sec. III except that $P_\Gamma = Q$ is now regarded as a fixed external parameter. Finally, one can reduce the action by solving the constraints as in Sec. IV. The result is again the action given by Eqs. (7.1) and (7.2).

Quantization of the reduced Hamiltonian theory proceeds as in Sec. V. For the renormalized trace of the analytically continued time evolution operator, we obtain

$$Z_{ren}(\beta,\mathbf{q}) = \int_{R_{crit}(\mathbf{q})}^\infty \tilde{\tilde{\mu}} dR_h \exp(-I_{C*}), \tag{7.3}$$

where the function R_{crit} is defined by Eq. (A4) in Appendix A, the weight factor $\tilde{\tilde{\mu}}$ is a positive function of R_h (and possibly $\mathbf{q}$), and

$$I_{C*}(R_h) := \tfrac{1}{2}\beta R_h(R_h^2 \ell^{-2} + 1 + \mathbf{q}^2 R_h^{-2}) - \pi R_h^2. \tag{7.4}$$

Under the assumption that $\tilde{\tilde{\mu}}$ is slowly varying, the dominant contribution to Z_{ren} can be estimated by saddle point methods. The cases $\mathbf{q}=0$ and $\mathbf{q}\neq 0$ merit each a separate analysis.

Consider first the special case $\mathbf{q}=0$. The lower limit of the integral in Eq. (7.3) is then at $R_h=0$. The critical point structure of I_{C*} is identical to that of I_* (7.4) for $\phi=0$, and the locations of the critical points and the values of the action at these points can simply be read off from Sec. VI by setting $\phi=0$.

Consider from now on the generic case $\mathbf{q}\neq 0$. I_{C*} has one negative critical point, and from one to three positive critical points. The negative critical point is unphysical, but all the positive critical points lie in the physical domain $R_h > R_{crit}(\mathbf{q})$. As I_{C*} is decreasing at $R_h = R_{crit}(\mathbf{q})$ and tends to infinity as $R_h \to \infty$, the global minimum of I_{C*} in the domain $R_h > R_{crit}(\mathbf{q})$ is at a critical point. We can, therefore, concentrate on the positive critical points.

When $\beta^2 \geq \frac{3}{2}\pi^2\ell^2$, I_{C*} has only one positive critical point. When $\beta^2 < \frac{3}{2}\pi^2\ell^2$, the number of positive critical points is determined by the status of the double inequality

$$\frac{(1-3s)(1+s)}{36(1-s)^2} \leq \mathbf{q}^2\ell^{-2} \leq \frac{(1+3s)(1-s)}{36(1+s)^2}, \tag{7.5}$$

where

$$s := \sqrt{1 - \frac{2\beta^2}{3\pi^2\ell^2}}. \tag{7.6}$$

When Eq. (7.5) does not hold, there is only one positive critical point. When Eq. (7.5) holds as a genuine inequality, there are three positive critical points, and saturating the inequalities gives limiting cases where two of the three positive critical points merge. [Note that if $\beta^2 \leq \frac{4}{3}\pi^2\ell^2$, the leftmost expression in Eq. (7.5) is nonpositive, and the left hand side inequality is then necessarily genuinely satisfied.] Now, when only one positive critical point exists, this critical point is the global minimum. On the other hand, when three positive critical points exist, they constitute a local maximum between two local minima, and the global minimum can be at either of the local minima depending on the values of the parameters. For example, when the right hand side inequality in Eq. (7.5) is close to being saturated, the global minimum is at the local minimum with the larger value of R_h.

The critical points can be examined further by parametrizing β and $\mathbf{q}$ as

$$4\pi\ell\beta^{-1} = \frac{(u-v)[3(u^2+v^2)+1]}{u^2+v^2-uv}, \tag{7.7a}$$

$$\mathbf{q}^2\ell^{-2} = \frac{u^2v^2(3uv+1)}{u^2+v^2-uv}, \tag{7.7b}$$

where the parameters u and v satisfy $0<v<u$. The negative, unphysical critical point is then at $R_h = -\ell v$, and $R_h = \ell u$ gives a positive critical point.[4] The condition that only one positive critical point exists reads

$$\alpha(u) < v, \tag{7.8}$$

where $\alpha(u)$ is the unique solution to the equation

$$0 = 9u\alpha^3 - (6u^2+1)\alpha^2 + u(9u^2+2)\alpha - u^2 \tag{7.9}$$

in the interval $0<\alpha<u$. In this case the parametrization (7.7) is unique. When the inequality in Eq. (7.8) is reversed and three positive critical points exist, the parametrization (7.7) can be made unique by imposing the conditions

$$u < \frac{1}{\sqrt{3}}, \tag{7.10a}$$

$$v < \frac{\sqrt{(1+6u^2)(1-3u^2)} - (1-3u^2)}{9u}, \tag{7.10b}$$

which make $R_h = \ell u$ the local maximum. The two local minima are then at the roots of the quadratic equation

$$0 = 3(u^2+v^2-uv)(R_h/\ell)^2 - (u-v)(3uv+1)(R_h/\ell) + uv(3uv+1). \tag{7.11}$$

The global minimum is at the larger (smaller) local minimum when the inequality

$$0 < 12(6uv-1)(u^2+v^2-uv)^2 + (u-v)^2(3uv+1)[3(u^2+v^2)+1] \tag{7.12}$$

[4]We thank Bernard Whiting for suggesting this type of parametrization.

is satisfied (reversed).

It is of some interest to examine the behavior of the critical points in the limit $\mathbf{q}^2 \to 0$ with fixed β. When $\beta^2 > \frac{4}{3}\pi^2 \ell^2$, the above discussion shows that for sufficiently small $\mathbf{q}^2$ there exists only one positive critical point, and in the limit $\mathbf{q}^2 \to 0$ this critical point approaches zero as

$$R_h = |\mathbf{q}|[1 + 2\pi\beta^{-1}|\mathbf{q}| + O(\mathbf{q}^2 \ell^{-2})]. \tag{7.13}$$

When $\beta^2 < \frac{4}{3}\pi^2 \ell^2$, on the other hand, there exist three positive critical points for sufficiently small $\mathbf{q}^2$. In the limit $\mathbf{q}^2 \to 0$, the smallest positive critical point again approaches 0 as Eq. (7.13), whereas the two larger ones approach the two critical points of the case $\mathbf{q}=0$. In the limiting case $\beta^2 = \frac{4}{3}\pi^2 \ell^2$, the smallest of the three positive critical points once again approaches zero as Eq. (7.13), and the two larger ones merge into a $\mathbf{q}=0$ critical point that is not a local extremum. The limiting behavior is thus smooth, in spite of the changing number of critical points.

At any critical point, the value of the action can be written as

$$I^c_{C*} = -\frac{\pi R_h^2 (R_h^2 \ell^{-2} - 1 - 3\mathbf{q}^2 R_h^{-2})}{3R_h^2 \ell^{-2} + 1 - \mathbf{q}^2 R_h^{-2}}. \tag{7.14}$$

In the limit $\mathbf{q} \to 0$, this agrees with the expression given in Ref. [35].

We thus see that for generic values of the parameters, $Z_{\text{ren}}(\beta, \mathbf{q})$ can be approximated by $\exp[-I^{\text{min}}_{C*}]$, where I^{min}_{C*} stands for the value of I_{C*} at the critical point that is the global minimum. As in Sec. VI, this is consistent with what one would have expected just from the existence of (Lorentzian) black hole solutions under fixing the charge and the renormalized inverse Hawking temperature: such solutions exist precisely at the critical points of I_{C*}, and the values of $\mathbf{m}$ and $\mathbf{q}$ at these critical points are just the mass and charge parameters of the black hole. One may view the shifting of the global minimum of I_{C*} from one local minimum to the other as a thermodynamical phase transition.

We end this section with some brief remarks on the thermodynamics in the canonical ensemble. Recall that the formulas for the thermal expectation values for the energy and the electric potential read

$$\langle E \rangle = -\frac{\partial(\ln Z_{\text{ren}})}{\partial \beta}, \tag{7.15a}$$

$$\langle \phi \rangle = -\beta^{-1} \frac{\partial(\ln Z_{\text{ren}})}{\partial \mathbf{q}}. \tag{7.15b}$$

When a critical point of I_{C*} dominates, we obtain

$$\langle E \rangle \approx \mathbf{m}, \tag{7.16a}$$

$$\langle \phi \rangle \approx \frac{\mathbf{q}}{R_h}. \tag{7.16b}$$

These are, respectively, just the mass and the electric potential difference between the horizon and the infinity for the dominating classical solution. When the approximation (7.16a) is good, the positivity of the (constant $\mathbf{q}$) heat capacity, $C_{\mathbf{q}} = -\beta^2(\partial\langle E \rangle / \partial \beta)$, follows from the fact that the dominant critical point is a minimum of I_{C*} [24,25]. The positivity of $C_{\mathbf{q}}$ follows more generally, even when the saddle point approximation does not hold, by direct manipulations from the expression (7.3) and the assumption that $\tilde{\tilde{\mu}}$ is positive.

When the saddle point approximation is good, we have for the entropy the Bekenstein-Hawking result, $S = [1 - \beta(\partial/\partial\beta)](\ln Z_{\text{ren}}) \approx \pi(R_h)^2$.

VIII. CONCLUSIONS AND DISCUSSION

In this paper we have investigated the Hamiltonian dynamics and thermodynamics of spherically symmetric Einstein-Maxwell theory with a negative cosmological constant. We first set up a classical Lorentzian Hamiltonian theory in which the right end of the spacelike hypersurfaces is at the asymptotically anti–de Sitter infinity in an exterior region of a RNAdS black hole spacetime, and the left end of the hypersurfaces is at the bifurcation two-sphere of a nondegenerate Killing horizon. We then simplified the constraints by a canonical transformation, and we explicitly reduced the theory into an unconstrained Hamiltonian theory with two canonical pairs of degrees of freedom. The reduced theory was quantized by Hamiltonian methods, and a grand partition function for a thermodynamical grand canonical ensemble was obtained by analytically continuing the Schrödinger picture time evolution operator to imaginary time and taking the trace. The analytic continuation at the bifurcation two-sphere was done in a way motivated by the smoothness of Euclidean black hole geometries as in Ref. [32]. A similar analysis with minor modifications to the boundary conditions led to a partition function for a thermodynamical canonical ensemble. Both the canonical ensemble and the grand canonical ensemble turned out to be well defined, and we were able to find the conditions under which the (grand) partition function is dominated by a classical Euclidean solution.

Both thermodynamical ensembles exhibited a phase transition. In the grand canonical ensemble the transition occurs when the grand partition function ceases to be dominated by any classical Euclidean black hole solution, in close analogy with what happens in the spherically symmetric vacuum canonical ensemble with a finite boundary [14,24,25]. In the canonical ensemble this kind of a phase transition can occur only in the limit of a vanishing charge, whereas for nonvanishing charge there occurs a phase transition in which the dominating contribution to the partition function shifts from one classical Euclidean solution to another as the boundary data changes. In either ensemble, whenever the (grand) partition function is dominated by a classical solution, one recovers for the entropy the Bekenstein-Hawking value of one quarter of the horizon area.

The classical canonical transformation of Sec. III is a relatively straightforward generalization of the transformation that was found by Kuchař in the spherically symmetric vacuum Einstein theory under Kruskal-like boundary conditions [44]. When the classical equations of motion hold, our new canonical coordinates M and Q are simply the mass and charge parameters of the RNAdS solution. By (generalized) Birkhoff's theorem, the spacetime is uniquely characterized by these two parameters and the cosmological constant. The conjugate momenta, P_M and P_Q, carry the information about the embedding of the spacelike hypersurface in the spacetime and the electromagnetic gauge. Upon elimination of the constraints, we saw in Sec. IV that P_M and P_Q each gives rise to one unconstrained momentum in the reduced Hamiltonian theory. These reduced momenta are global constructs with no local geometrical meaning, and they are associated with the anchoring of the spacelike hypersurfaces at the infinity and at the bifurcation two-sphere. The electromagnetic pair (Q,P_Q) is quite closely analogous to the gravitational pair (M,P_M). The third canonical pair, $(\mathsf{R},P_{\mathsf{R}})$, is entirely gauge, and it completely disappears when the constraints are eliminated.

Although we have here discussed the canonical transformation only under boundary conditions motivated by our thermodynamical goal, it would appear possible to use arguments similar to those in Refs. [44,55] to adapt this canonical transformation to boundary conditions under which the spacelike hypersurfaces extend from a left-hand side asymptotically anti–de Sitter region to a right-hand side asymptotically anti–de Sitter region, crossing the event horizons in arbitrary ways. The form (3.15) taken by the constraints then suggests that, after introducing electromagnetic variables analogous to the reparametrization clocks $\tau_\pm$ of Ref. [44], it is possible to perform a canonical transformation that separates Q into the charge density Q' and the charge at the (say) left-hand side infinity, in analogy with the transformation that in Ref. [44] separates M into the mass density M' and the mass at the left-hand side infinity. Also, it appears possible to take the limit where the cosmological constant vanishes and the asymptotically anti–de Sitter regions are replaced by asymptotically flat regions.[5] It would further be possible to consider boundary conditions of the kind put forward in Ref. [57]. We have not investigated these issues in a systematic fashion; however, we shall outline in Appendix B how our canonical transformation can be adapted to the limit of a vanishing cosmological constant, under boundary conditions that still keep the left end of the hypersurfaces at the bifurcation two-sphere of a nondegenerate Killing horizon but replace the asymptotically anti–de Sitter falloff conditions at the right end by asymptotically flat falloff conditions. In this case, each classical solution is the exterior region of a nonextremal Reissner-Nordström black hole.

The thermodynamical results of Secs. VI and VII show that the stabilizing effect of the negative cosmological constant is highly similar to the stabilizing effect of a finite ''box'' with fixed surface area and fixed local temperature [14,24–26,34]. One important difference is, however, that in the asymptotically anti-de Sitter case various thermal expectation values are more directly related to the parameters of the dominant classical solutions. In the grand canonical ensemble, Eqs. (6.9) show that the thermal expectation values of energy and charge are simply the mass and charge parameters of the dominant classical solution: there is no additional contribution to the mass from the gravitational binding energy associated with the thermal energy, or from the electrostatic binding energy associated with the charge. Such additional, finite-size contributions were found to be present in the finite-size ensembles considered in Refs. [14,24,26,34]. In the canonical ensemble, the situation is similar with the thermal expectation values of the energy and the electric potential (7.16).

The stabilizing effect of the negative cosmological constant becomes fully apparent when one attempts to repeat the analysis with a vanishing cosmological constant, replacing the asymptotically anti–de Sitter infinity by an asymptotically flat infinity. We shall outline this analysis in Appendix B. While there is no difficulty in quantizing the reduced Hamiltonian theory, the trace of the analytically continued time evolution operator turns out to remain divergent even after a renormalization of the kind performed in Secs. V B and VII. Neither the canonical ensemble nor the grand canonical ensemble exists. For the canonical ensemble this conclusion might be surprising in view of the observation that a Reissner-Nordström black hole in asymptotically flat space is stable against Hawking evaporation when one fixes the charge and the temperature at the infinity, provided the mass and charge parameters of the hole satisfy the inequality $\mathbf{q}^2 > \frac{3}{4}\mathbf{m}^2$ [7]. However, as we shall see in Appendix B, the local stability of a classical solution is not sufficient to guarantee the existence of a full thermodynamical ensemble.

Finally, we recall that as the physical temperature of Hawking radiation is redshifted to zero at the anti–de Sitter infinity, we followed Refs. [18,35,36] and defined a renormalized temperature at infinity in terms of the rate at which the local Hawking temperature approaches zero. This definition led to physically reasonable conclusions; in particular, we recovered from the thermodynamical ensembles the Bekenstein-Hawking result for the black hole entropy. The definition can however be argued to have an *ad hoc* flavor, and one might wish to replace it by something that can be given a more immediate physical justification.[6] What would be needed is a better understanding as to whether asymptotically anti–de Sitter infinity can in some sense be regarded as a physically realizable system, rather than just as a mathematically elegant set of boundary conditions.

Note added in proof. After this work was completed, we became aware of Refs. [58–60], which discuss the Dirac quantization of four-dimensional spherically symmetric Einstein-Maxwell geometries and related dilatonic theories. The work in these references has close technical similarities to our work.

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[5]Before the work reported in this paper was begun, we were informed by Karel Kuchař that he had generalized the canonical transformation of Ref. [44] to the spherically symmetric Einstein-Maxwell system without a cosmological constant [56]. We thank Karel Kuchař for correspondence on this point.

[6]An interesting possibility might be to *assume* the Bekenstein-Hawking entropy and then to derive the appropriate renormalized temperature [38]. However, it does not appear clear how to adopt this as a starting point in a theory where the Bekenstein-Hawking result is expected to be only an approximate one, in the domain where the (grand) partition function is dominated by a classical Euclidean solution.

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APPENDIX A: REISSNER–NORDSTRÖM–ANTI-de SITTER BLACK HOLE

In this appendix we recall some relevant properties of the Reissner–Nordström–anti-de Sitter (RNAdS) metric. We concentrate on the case where a nondegenerate event horizon exists, and on the region exterior to this horizon.

In the curvature coordinates (T,R), the RNAdS metric is given by

$$ds^2 = -F dT^2 + F^{-1} dR^2 + R^2 d\Omega^2, \tag{A1a}$$

where $d\Omega^2$ is the metric on the unit two-sphere and

$$F := \frac{R^2}{\ell^2} + 1 - \frac{2M}{R} + \frac{Q^2}{R^2}. \tag{A1b}$$

T and R are called, respectively, the Killing time and the curvature radius. The parameter ℓ is positive, and we take the parameters M and Q to be real. Together with the electromagnetic potential

$$A = \frac{Q}{R} dT, \tag{A2}$$

the metric (A1) is a solution to the Einstein-Maxwell equations with the cosmological constant $-3\ell^{-2}$ [40,43]. The parameters M and Q are referred to, respectively, as the mass and the (electric) charge. The case $Q=0$ yields the Schwarzschild–anti-de Sitter metric, and the case $Q=M=0$ yields the metric on (the universal covering space of) anti–de Sitter space [61].

The metric (A1) has an asymptotically anti–de Sitter infinity at $R\to\infty$ for all values of the parameters [47]. We wish to restrict the parameters so that the metric describes the exterior of a black hole with a nondegenerate horizon. This happens when the quartic polynomial $R^2F(R)$ has a simple positive root $R=R_0$, such that F is positive for $R>R_0$. The necessary and sufficient condition is $M>M_{\rm crit}(Q)$, where

$$M_{\rm crit}(Q) := \frac{\ell}{3\sqrt{6}}\left(\sqrt{1+12(Q/\ell)^2}+2\right) \times \left(\sqrt{1+12(Q/\ell)^2}-1\right)^{1/2}. \tag{A3}$$

Note that M is then necessarily positive. R_0 can now be determined uniquely as the function $R_{\rm hor}(M,Q)$ of M and Q: for $Q=0$, $R_{\rm hor}(M,Q)$ is defined as the unique positive solution to the equation $F=0$; for $Q\neq 0$, $R_{\rm hor}(M,Q)$ is defined as the larger of the two positive solutions. In either case, if Q is considered fixed, $R_{\rm hor}(M,Q)$ is a monotonically increasing function of M that takes the values $R_{\rm crit}(Q)<R_{\rm hor}(M,Q)<\infty$ as $M_{\rm crit}(Q)<M<\infty$, where

$$R_{\rm crit}(Q) := \frac{\ell}{\sqrt{6}}\left(\sqrt{1+12(Q/\ell)^2}-1\right)^{1/2}. \tag{A4}$$

The metric can thus be uniquely parametrized by Q and R_0. The only restriction for these parameters is

$$R_0 > R_{\rm crit}(Q), \tag{A5}$$

and the mass is then given by

$$M = \frac{R_0}{2}\left(\frac{R_0^2}{\ell^2} + 1 + \frac{Q^2}{R_0^2}\right). \tag{A6}$$

With $R_0<R<\infty$, the metric (A1a) covers the region from the horizon to the asymptotically anti–de Sitter infinity. The Penrose diagram can be found in Refs. [37,62].

APPENDIX B: REISSNER-NORDSTRÖM BLACK HOLE IN ASYMPTOTICALLY FLAT SPACE

In the main text we took the cosmological constant to be strictly negative. In this appendix we shall outline the corresponding classical and quantum-mechanical analyses in the case where the cosmological constant vanishes. In the notation of the main text this means taking the limit $\ell\to\infty$. The classical solutions are then not asymptotically anti–de Sitter but asymptotically flat, and the falloff conditions at $r\to\infty$ must be modified to reflect this fact.

In the variables of Sec. II, we retain the falloff conditions (2.11) at $r\to 0$, but at $r\to\infty$ we introduce the new falloff conditions

$$\Lambda(t,r) = 1 + M_+(t) r^{-1} + O^\infty(r^{-1-\epsilon}), \tag{B1a}$$

$$R(t,r) = r + O^\infty(r^{-\epsilon}), \tag{B1b}$$

$$P_\Lambda(t,r) = O^\infty(r^{-\epsilon}), \tag{B1c}$$

$$P_R(t,r) = O^\infty(r^{-1-\epsilon}), \tag{B1d}$$

$$N(t,r) = N_+(t) + O^\infty(r^{-\epsilon}), \tag{B1e}$$

$$N^r(t,r) = O^\infty(r^{-\epsilon}), \tag{B1f}$$

$$\Gamma(t,r) = O^\infty(r^{-1-\epsilon}), \tag{B1g}$$

$$P_\Gamma(t,r) = Q_+(t) + O^\infty(r^{-\epsilon}), \tag{B1h}$$

$$\tilde\Phi(t,r) = \tilde\Phi_+(t) + O^\infty(r^{-\epsilon}), \tag{B1i}$$

where $0<\epsilon\leq 1$. For the metric quantities these conditions are precisely those used in Ref. [44], ensuring asymptotic flatness. These conditions make the bulk action (2.6) well defined, and they are preserved under the time evolution. Adding the boundary action

$$\int dt\left(\tfrac{1}{2}R_0^2 N_1 \Lambda_0^{-1} - N_+ M_+ + \tilde\Phi_0 Q_0 - \tilde\Phi_+ Q_+\right) \tag{B2}$$

yields an action for a variational principle in which N_+, $N_1\Lambda_0^{-1}$, $\tilde\Phi_+$, and $\tilde\Phi_0$ are prescribed functions of t. Dropping the last two terms in Eq. (B2) yields an action for a variational principle in which $\tilde{\Phi}_+$ and $\tilde{\Phi}_0$ are free but Q_+ and Q_0 are prescribed.

The canonical transformation of the main text can now be adapted to the present boundary conditions by simply taking the limit $\ell\to\infty$. A new action can be constructed as in Sec. III C, the new falloff conditions only giving rise to minor technical modifications to the redefinition of the Lagrange multipliers. In the theory that prescribes $\tilde{\Phi}_+$ and $\tilde{\Phi}_0$, elimination of the constraints along the lines of Sec. IV yields the reduced action

$$S[\mathbf{m},\mathbf{q},\mathbf{p_m},\mathbf{p_q};\tilde{N}_0^M,N_+,\tilde{\Phi}_+,\tilde{\Phi}_0]=\int dt(\mathbf{p_m}\dot{\mathbf{m}}+\mathbf{p_q}\dot{\mathbf{q}}-\mathbf{h}), \tag{B3}$$

where the reduced Hamiltonian is given by

$$\mathbf{h}=-\tfrac{1}{2}R_{\mathrm{h}}^2\tilde{N}_0^M+N_+\mathbf{m}+(\tilde{\Phi}_+-\tilde{\Phi}_0)\mathbf{q} \tag{B4}$$

with $R_{\mathrm{h}}:=\mathbf{m}+\sqrt{\mathbf{m}^2-\mathbf{q}^2}$. The range of the variables is $0<\mathbf{m}$, $\mathbf{q}^2<\mathbf{m}^2$. In the theory that prescribes Q_+ and Q_0, one proceeds as in Sec. VII to obtain the reduced action

$$S_C[\mathbf{m},\mathbf{p_m};\tilde{N}_0^M,N_+;\mathbf{q}]=\int dt(\mathbf{p_m}\dot{\mathbf{m}}-\mathbf{h}_C), \tag{B5}$$

where $\mathbf{q}$ is now regarded as an external parameter and

$$\mathbf{h}_C=-\tfrac{1}{2}R_{\mathrm{h}}^2\tilde{N}_0^M+N_+\mathbf{m}. \tag{B6}$$

Quantization of the two reduced theories proceeds as in the main text. For the renormalized trace of the analytically continued time evolution operator, we obtain formally

$$\mathcal{Z}_{\mathrm{ren}}(\beta,\phi)=\mathcal{N}\int_{R_{\mathrm{h}}>|\mathbf{q}|}\tilde{\mu}\,dR_{\mathrm{h}}\,d\mathbf{q}\exp(-I_*), \tag{B7a}$$

$$Z_{\mathrm{ren}}(\beta,\mathbf{q})=\int_{|\mathbf{q}|}^{\infty}\tilde{\tilde{\mu}}\,dR_{\mathrm{h}}\exp(-I_{C*}), \tag{B7b}$$

where I_* and I_{C*} are, respectively, given by dropping the term proportional to ℓ^{-2} from Eqs. (6.3) and (7.4). β is now interpreted as the inverse Hawking temperature at the infinity, with no renormalization. However, both integrals in Eqs. (B7) are divergent because of the behavior of I_* and I_{C*} at large R_{h}. Thus, neither the canonical ensemble nor the grand canonical ensemble exists under the asymptotically flat boundary conditions. In this respect, the inclusion of the charge has, therefore, not made a qualitative difference from the asymptotically flat vacuum case [32].

The critical points of I_* and I_{C*} give again the (Lorentzian) classical solutions that have the inverse Hawking temperature β at infinity and the prescribed value of, respectively, ϕ or $\mathbf{q}$. The condition that I_* possesses critical points is $|\phi|<1$: when this condition is satisfied there exists exactly one critical point, but this critical point is not a local extremum. This reproduces the observations made by Davies in Refs. [7,8] about charged black hole equilibria with fixed ϕ, and reflects, in particular, the fact that a semiclassical charged black hole under these boundary conditions is not stable against Hawking evaporation.

The condition that I_{C*} possesses critical points is $\beta/|\mathbf{q}|\geqslant 6\pi\sqrt{3}$, and when the inequality is genuine, there exist two critical points. The critical point with the smaller (larger) value of R_{h} is a local minimum (maximum, respectively). The local minimum satisfies $\mathbf{q}^2>\frac{3}{4}\mathbf{m}^2$, and corresponds to the classical solution that Davies [7] showed to be stable against Hawking evaporation under these boundary conditions (see also Refs. [63–65]). While the thermodynamical stability of this semiclassical solution is reflected in its being a local minimum of our I_{C*} [25], the divergence of the integral in Eq. (B7b) demonstrates that this local stability is not sufficient to guarantee the existence of a full thermodynamical canonical ensemble.

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